Problem Set 1

1. (This exercise will be solved in the first exercise class.)

Let $p \in (1, \infty)$, $\Omega \subseteq \mathbb{R}^n$ a bounded domain, $f \in L^2(\Omega)$ and define the functional

$$E\colon L^p(\Omega) \cap H^1(\Omega) \to \mathbb{R},\ u \mapsto \int_\Omega \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p}|u|^p + fu\,dx.$$

Show that $E \in C^1(L^p(\Omega) \cap H^1(\Omega))$.

2. Let $p \in (1, \infty)$, $\Omega \subseteq \mathbb{R}^n$ a bounded domain, $n \geq 2$, $f \in L^2(\Omega)$ and define the functional

$$E\colon H_0^1(\Omega) \to \mathbb{R} \cup \{\pm\infty\},\ u \mapsto \int_\Omega \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p}|u|^p + fu\,dx.$$

For which p is $E \in C^1(H_0^1(\Omega))$?

3. Let $\Omega \subseteq \mathbb{R}^n$ be bounded and let $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n$, $(x, q, p) \mapsto f(x, q, p)$ be a smooth function. Define the functional

$$E\colon C^1(\bar{\Omega}) \to \mathbb{R},\ u \mapsto \int_\Omega f(x, u(x), \nabla u(x))dx.$$

Suppose $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is a minimizer of E with respect to compactly supported variations in the sense that $E(u) \leq E(u + \varphi)$ for all $\varphi \in C_c^1(\Omega)$.

(a) Show that for this minimizing u the Euler-Lagrange equation holds:

$$\frac{\partial f}{\partial q}(x, u(x), \nabla u(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial p_i}(x, u(x), \nabla u(x))\right) = 0.$$

(b) Let $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Derive sufficient conditions on a and b such that

$$-\operatorname{div}(a(x, u(x), \nabla u(x))) + b(x, u(x), \nabla u(x)) = 0$$

is the Euler-Lagrange equation of a functional as defined above.