Problem Set 10

1. Präsenzaufgabe: Pseudo-Gradient Vector Field. Let $S \subseteq \mathbb{R}^3$ be diffeomorphic to $\mathbb{S}^2$ and let $\pi \colon U_d S \to S$ be the nearest neighbour projection. Let

$$M = H^1(\mathbb{S}^1, S) := \{u \in H^1(\mathbb{S}^1, \mathbb{R}^3) \mid u(t) \in S \text{ for a.e. } t\}.$$

Let $E \in C^1(M)$ be any functional and $\tilde{M}$ the regular values of E. A pseudo-gradient vector field for E is $e \colon \tilde{M} \to H^1(\mathbb{S}^1, \mathbb{R}^3)$ locally Lipschitz continuous such that $e(u) \in T_u M$ and satisfying for all $u \in \tilde{M}$:

$$\begin{aligned} \|e(u)\|_{T_u M} &< 1 \\ \langle dE(u), e(u)\rangle_{T_u^* M \times T_u M} &> \tfrac{1}{2}\|dE(u)\|. \end{aligned}$$

(a) Use the map $d\pi(u) \colon H^1(\mathbb{S}^1, \mathbb{R}^3) \to T_u M$ to construct a pseudo-gradient vector field.

(b) Does this construction also work for an orientable closed submanifold $S \subseteq \mathbb{R}^n$, $n \geq 4$?

2. Flow Invariant Family. In the lecture we have seen that on any $S \subseteq \mathbb{R}^3$ which is diffeomorphic to the sphere $\mathbb{S}^2$ there is a nonconstant closed geodesic. For this we constructed a flow-invariant family $\mathcal{F}$, which consisted of 1-parameter families of closed curves inducing a map $p \colon \mathbb{S}^2 \to \mathbb{S}^2$ homotopic to the identity.

Assume now $S \subseteq \mathbb{R}^n$ is diffeomorphic to $\mathbb{S}^{n-1}$. Find a similar family $\mathcal{F}$ and prove it is flow-invariant under any continuous 1-parameter family of homeomorphisms which maps constant paths to constant paths.

Hint: The $(n-1)$-dimensional sphere can be parametrized by $\varphi \in [0, 2\pi]$, $\vartheta_i \in [0, \pi]$ via

$$\begin{aligned} \cos\varphi \sin\vartheta_1 \cdots \sin\vartheta_{n-2} &= x_1 \\ \sin\varphi \sin\vartheta_1 \cdots \sin\vartheta_{n-2} &= x_2 \\ \cos\vartheta_1 \cdots \sin\vartheta_{n-2} &= x_3 \\ &\vdots \\ \cos\vartheta_{n-2} &= x_n. \end{aligned}$$

3. Closed Geodesic. Use Exercise 2 to characterise a non-constant closed geodesic on S by a minimax-argument.