Problem Set 11

1. Category. Let M be a smooth, connected, compact n-dimensional manifold without boundary and $A \subseteq M$ a (topologically) closed subset. We say that A has category k relative to M, written $\operatorname{cat}_M(A) = k$, if A can be covered by k closed sets U_i, $i = 1, \dots, k$, which are contractible in M and if k is minimal with this property. If there is no such finite k we define $\operatorname{cat}_M(A) = \infty$ and $\operatorname{cat}_M(\emptyset) = 0$. Show that

(a) $\operatorname{cat}_{\mathbb{S}^2}(\mathbb{S}^2) = 2,$

(b) $\operatorname{cat}_{\mathbb{T}^2}(\mathbb{T}^2) = 3,$ where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the torus,

(c) $\operatorname{cat}_{\mathbb{RP}^2}(\mathbb{RP}^2) = 3,$ where $\mathbb{RP}^2 = \mathbb{S}^2/\{\pm 1\}$ is the projective space.

2. Index Let $\mathcal{A} \subseteq \mathcal{P}(M)$ be the set of all closed subsets of a manifold M as in Exercise 1.

(a) Show that $\operatorname{cat}_M \colon \mathcal{A} \to \mathbb{Z}$ defines an index, i.e. satisfies the following properties for all $A, A_1, A_2 \in \mathcal{A}$ and all homeomorphisms $h \colon M \to M$:

i) $\operatorname{cat}_M(A) \geq 0$ and $\operatorname{cat}_M(A) = 0$ if and only if $A = \emptyset$.

ii) If $A_1 \subseteq A_2$, then $\operatorname{cat}_M(A_1) \leq \operatorname{cat}_M(A_2)$.

iii) $\operatorname{cat}_M(A_1 \cup A_2) \leq \operatorname{cat}_M(A_1) + \operatorname{cat}_M(A_2)$.

iv) $\operatorname{cat}_M(A) \leq \operatorname{cat}_M(h(A))$.

v) If A is compact, then $\operatorname{cat}_M(A) < \infty$ and there exists an open neighbourhood $A \subseteq N$ with $\operatorname{cat}_M(\overline{N}) = \operatorname{cat}_M(A)$.

vi) If A is finite, then $\operatorname{cat}_M(A) = 1$.

(b) Show that any $f \in C^1(M)$ admits at least $\operatorname{cat}_M(M)$ critical points.

3. Billard. Let $f \in C^\infty(\mathbb{R}^2)$ satisfy

$$\begin{aligned}
&f(s,t) = f(s,t+1) = f(s+1,t),\\
&f(s,t) = f(t,s),\\
&f(s,s) = 0,\\
&f(s,t) > 0, \quad \text{if } s \neq t.
\end{aligned}$$

In view of this symmetries we may regard $D = \{(s,t) \mid 0 \leq s \leq t \leq 1\}$ as fundamental domain for f.

Construct $\varphi \colon D \to \mathbb{RP}^2$ and $\tilde{f} \colon \mathbb{RP}^2 \to \mathbb{R}$ continuous such that $f = \tilde{f} \circ \varphi$. Conclude, using the category defined in Exercise 1, that $\tilde{f}$ has at least three critical points.