Problem Set 2

1. Präsenzaufgabe. Consider the functional E introduced in the lecture:

$$E(u) = \int_0^d (\sqrt{1 + |u'|^2} + u)\, dx,$$

where

$$u \in M := \{u \in W^{1,1}([0,d]) \mid u(0) = 0,\, u(d) = 1\}.$$

Show that

(a) E is coercive if $d < 2$,

(b) $\inf_{u \in M} E(u) = -\infty$ if $d > 2$,

(c) E is bounded from below but not coercive if $d = 2$.

2. Minimisation of General Functionals. Let $\Omega \subseteq \mathbb{R}^n$ be bounded domain and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $(x, s, p) \mapsto f(x, s, p)$ be a function which is continuous in s and convex in p. Assume further that there exists a constant C such that

$$|p|^2 \le f(x, s, p) \le C|p|^2 + C$$

for all (x, s, p). Prove that the functional

$$\begin{aligned} E : H_0^1(\Omega) &\to \mathbb{R} \\ u &\mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx \end{aligned}$$

attains its infimum.

3. Weak maximum principle. Let $\Omega \subseteq \mathbb{R}^n$ be a domain of class C^2 and let L be the operator defined by $Lu := -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a^{ij} \frac{\partial}{\partial x_j} u\right) + cu$, where $a^{ij} \in C^1(\bar{\Omega})$ and $(a^{ij}(x))$ is symmetric, uniformly positive definite with respect to $x \in \Omega$ and $c \in C^0(\bar{\Omega})$, $c \ge 0$, i.e. L is uniformly elliptic.

Definition. $u \in H^1(\Omega)$ is a weak subsolution (written $Lu \le 0$), if

$$\sum_{i,j} \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \varphi}{\partial x_j} + cu\varphi\, dx \le 0$$

for all $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$.

(a) Let u be a weak subsolution and show that if $u \le 0$ on $\partial\Omega$, then $u \le 0$ on all of Ω.

(b) Assume $u \in H^1(\Omega)$ is a solution of $Lu = 0$. Show that

$$\sup_\Omega |u| = \sup_{\partial\Omega} |u|.$$

Hand in the solution by 8th October 2014