Problem Set 3

1. Strong Maximum Principle Let $\Omega \subseteq \mathbb{R}^n$ be a domain of class C^2 and let L be the operator defined by $Lu := -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial}{\partial x_j}u\right) + cu$, where $a^{ij} \in C^1(\bar{\Omega})$ and $(a^{ij}(x))$ is symmetric, uniformly positive definite with respect to $x \in \Omega$ and $c \in C^0(\bar{\Omega})$, $c \geq 0$, i.e. L is uniformly elliptic.

Let $u \in C^1(\bar{\Omega})$ be a weak subsolution. Assume $u \leq M$ on $\partial\Omega$ for some $M \geq 0$, hence $u \leq M$ on all of Ω by the weak maximum principle. Assume further $u(x_0) = M$ for some $x_0 \in \Omega$. Then conclude $u = M$ in all of Ω.

2. Let L be as in exercise 1, where now $c \in C^0(\Omega)$ is arbitrary.

(a) Show that if $u \in H_0^1(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Lu \leq 0$ in Ω, $u \leq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$.

(b) Is the analogue of exercise 3(b) from Problem Set 2 true in this case, i.e. do we again have the estimates

$$\sup_{\Omega}|u| = \sup_{\partial\Omega}|u|?$$

3. Consider the problem

$$\begin{cases} -\Delta u = u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $u_1, u_2 \in H_0^1(\Omega) \cap C^1(\bar{\Omega})$ are both solutions to this problem satisfying $u_1 \leq u_2$. Show that either $u_1 \equiv u_2$ or $u_1 < u_2$ everywhere.

4. Assumme $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $u \in H_0^1(\Omega) \cap C^2(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = u|u|^{p-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $u > 0$ in Ω. Show that $\frac{\partial u}{\partial \nu}$ cannot vanish on all of $\partial\Omega$.

Hand in the solution by 15th October 2014.