Problem Set 4

1. (This exercise will be solved in the exercise class.)

Let $n \geq 3$, $a \in C^0(\mathbb{R}^n)$ and assume $a(x) \to a_\infty > 0$, $|x| \to \infty$. Recall the functional

$$E\colon H^1(\mathbb{R}^n) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + a(x)|u(x)|^2\right) dx$$

and the functional at infinity

$$E_\infty\colon H^1(\mathbb{R}^n) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + a_\infty |u(x)|^2\right) dx$$

on $M := \{u \in H^1(\mathbb{R}^n) \mid \|u\|_{L^p} = 1\}$, where $2 < p < 2^*$, as well as

$$I := \inf_{u \in M} E(u), \quad I_\infty := \inf_{u \in M} E_\infty(u).$$

We proved that $I < I_\infty$ is necessary and sufficient for the convergence of any minimzing sequence $(u_k) \subseteq M$ for E.

Assume now that $a(x) < a_\infty$ for all $x \in \mathbb{R}^n$ and prove that under this assumption $I < I_\infty$ holds.

2. Let $a \in C^0(\mathbb{R}^n)$ and suppose $a(x) \to a_\infty \in \mathbb{R}$ as $|x| \to \infty$. (No assumption on the sign of a_∞.) Let $2 < p < \frac{2n+4}{n} < 2^*$ and consider the functional

$$E\colon H^1(\mathbb{R}^n) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 - a(x)|u(x)|^p\right) dx$$

as well as the functional at infinity

$$E_\infty\colon H^1(\mathbb{R}^n) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 - a_\infty |u(x)|^p\right) dx.$$

Let

$$M_\lambda = \{u \in H^1(\mathbb{R}^n) \mid \|u\|^2_{L^2(\mathbb{R}^n)} = \lambda\}$$

and define

$$I_\lambda := \inf_{u \in M_\lambda} E(u), \quad I_{\lambda,\infty} := \inf_{u \in M_\lambda} E_\infty(u).$$

(a) Show that E is coercive on M_λ by showing an inequality of the type $\|u\|_p^p \leq C\|\nabla u\|_2^\beta$, where $0 < \beta < 2$.

(b) Show that for all $\lambda \in [0, \infty)$ and for all $\alpha \in [0, \lambda]$ we have the inequality

$$I_\lambda \leq I_{\alpha,\infty} + I_{\lambda-\alpha}. \tag{1}$$

(c) Assume equality in (1) for some $\lambda > 0$ and some $\alpha \in (0, \lambda]$. Find a minimizing sequence for E on M_λ which is not relatively compact.

(d) Let $\lambda > 0$. Prove that strict inequality in (1) for all $\alpha \in (0, \lambda]$ is equivalent to relative compactness of all minimizing sequences for E on M_λ.

(e) Use the result from part (d) to show that if $a_\infty \leq 0$, then relative compactness of all minimizing sequences for E on M_λ is equivalent to $I_\lambda < 0$.

Hand in the solution by 22nd October 2014.