Problem Set 5

1. Concentration – Compactness. Consider the functional

$$E\colon L^1(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$$
$$u \mapsto \int_{\mathbb{R}^n} u(x)^2\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)u(y)\,\chi_{B_R(0)}(x-y)\,dx\,dy,$$

where $\chi_{B_R(0)}$ is the characteristic function of the ball of radius R. For $\lambda > 0$ let

$$M_\lambda := \{u \in L^1(\mathbb{R}^n) \mid u \geq 0 \text{ a.e. and } \int_{\mathbb{R}^n} u\,dx = \lambda\}$$

and define

$$I_\lambda := \inf_{u \in M_\lambda} E(u).$$

(a) Show that the condition

$$\forall \lambda \in (0,1): \quad I_1 < I_\lambda + I_{1-\lambda} \tag{1}$$

is equivalent to the condition

$$\exists \lambda > 0: \qquad I_\lambda < 0. \tag{2}$$

(b) Assume (1) holds. Show that then there exists a minimium of E on M_1.

(c) Show that $I_\lambda < 0$ for R sufficiently large.

2. A Variant. What happens if we consider the functional

$$E\colon H^1 \cap L^1(\mathbb{R}^n) \to \mathbb{R}$$
$$u \mapsto \int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)u(y)\,\chi_{B_R(0)}(x-y)\,dx\,dy$$

and ask the same questions as in Exercise 1?