Problem Set 6

1. Brezis-Nirenberg: A Baby Yamabe Problem. Let $n \geq 3$ and $\Omega \subseteq \mathbb{R}^n$ a bounded domain of class C^2. Consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = u|u|^{2^*-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The corresponding functional one needs to minimise is

$$\begin{aligned} E_\lambda \colon H_0^1(\Omega) &\to \mathbb{R} \\ u &\mapsto \int_\Omega |\nabla u(x)|^2 - \lambda |u(x)|^2 \, dx, \end{aligned}$$

restricted to

$$M := \{u \in H_0^1(\Omega) \mid \|u\|_{2^*} = 1\}.$$

Define

$$\begin{aligned} S_\lambda &:= \inf_{0 \neq u \in C_c^\infty(\Omega)} \frac{E_\lambda(u)}{\|u\|_{2^*}^2}, \\ \lambda_1 &:= \inf_{0 \neq u \in C_c^\infty(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} > 0. \end{aligned}$$

The first Dirichlet eigenvalue λ_1 is indeed positive, see Satz and Bemerkung 9.4.1. from M. Struwe: FAII as a reference.

(a) Show that E_λ is coercive for $\lambda < \lambda_1$.

(b) Show that for all $\lambda \in \mathbb{R}$ it holds $S_\lambda \leq S_0 =: S$ and $S_\lambda = S$ for $\lambda \leq 0$.

(c) Show that $S_\lambda < S$ for $\lambda < \lambda_1$ close enough to λ_1.

(d) Show that for $\lambda < \lambda_1$ the condition $S_\lambda < S = S_0$ is necessary and sufficient for that each minimising sequence for E_λ on M has a weak H_0^1-converging subsequence with limit in M.