Problem Set 7

1. Hodge $*$-Operator. Recall from the lecture that we defined the Hodge $*$-operator on $\mathbb{R}^n$ by

$$\begin{aligned} *\,1 &= dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \\ *(a \cdot b) &= a \wedge * b, \quad \text{for } k\text{-forms } a, b, \end{aligned}$$

where for $a = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $b = \sum_{1 \le i_1 < \ldots < i_k \le n} b_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ the inner product is

$$a \cdot b = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1,\ldots,i_k}\, b_{i_1,\ldots,i_k}.$$

Define the Hodge differential d^* as $d^*\omega = *\,d * \omega$ and the Hodge Laplacian as $\Delta = d^*d + dd^*$.

(a) Let $u \in W^{1,n}(\Omega; \mathbb{R}^n)$, where $\Omega \subseteq \mathbb{R}^n$. Show that $*\det(du) = du^1 \wedge \ldots \wedge du^n$.

(b) Let $a = \sum_{i=1}^n a_i\, dx^i$ be a 1-form. Show that d^*a equals the divergence of the corresponding vector field $(a_1, \ldots, a_n)\colon \Omega \to \mathbb{R}^n$.

(c) Let $f \in C^\infty(\Omega)$. Show that $(d^*d + dd^*)f = \operatorname{div} \nabla f$, the usual Laplacian of f.

2. Area Functional. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Consider the functional

$$\begin{aligned} E\colon W^{1,1}(\Omega) &\to \mathbb{R} \\ u &\mapsto \int_\Omega \sqrt{1 + |\nabla u(x)|^2} - 1\, dx. \end{aligned}$$

The Gateaux derivative of E is given by

$$dE(u)v = \int_\Omega \frac{\nabla u(x) \cdot \nabla v(x)}{\sqrt{1 + |\nabla u(x)|^2}}\, dx.$$

(a) Show that E is not Fréchet differentiable at any $u_0 \in W^{1,1}(\Omega)$.

(b) Show that the map

$$\begin{aligned} W^{1,1}(\Omega) &\to (W^{1,1}(\Omega))^* \\ u &\mapsto dE(u) \end{aligned}$$

is not continuous.