Problem Set 8

1. Präsenzaufgabe. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $2 < p < 2^*$. Consider the functional

$$\begin{aligned} E\colon H_0^1(\Omega) &\to \mathbb{R} \\ u &\mapsto \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \frac{1}{p}\int_\Omega |u(x)|^p\,dx. \end{aligned}$$

We have already seen that $E \in C^1(H_0^1(\Omega))$ and that E satisfies the Palais-Smale condition. Let $\bar{u} \in H_0^1(\Omega)$ with $\|\bar{u}\|_p = 1$ satisfy

$$\|\nabla \bar{u}\|_2^2 = \inf_{v\in M} \|\nabla v\|_2^2 =: \alpha,$$

where

$$M := \{v \in H_0^1(\Omega) \mid \|v\|_p = 1\}.$$

(a) Let

$$\Gamma = \{\gamma \in C^0([0,1]; H_0^1(\Omega)) \mid \gamma(0) = 0,\ E(\gamma(1)) < 0\}.$$

Show that for each $\gamma \in \Gamma$ there is some $0 < s_\gamma < 1$ such that for $w = \gamma(s_\gamma)$ we have the property

$$\langle dE(w), w\rangle = 0,$$

which is equivalent to the condition

$$\|\nabla w\|_2^2 - \|w\|_p^p = 0.$$

(b) Let $v \in M$. Compute $\sup\limits_{0<\lambda<\infty} E(\lambda v)$.

(c) Prove the identity

$$\sup_{0<\lambda<\infty} E(\lambda \bar{u}) = \inf_{\gamma\in\Gamma} \sup_{0\le s\le 1} E(\gamma(s)) =: \beta.$$

(d) Let $u \in K_\beta$, where $K_\beta = \{u \in H_0^1(\Omega) \mid E(u) = \beta,\ dE(u) = 0\}$. Show that $\tilde{u} := \frac{u}{\|u\|_p} \in M$ satisfies

$$\|\nabla \tilde{u}\|_2^2 = \alpha.$$

2. Cerami. Let X be a Banach space and $E \in C^1(X)$. Let $\beta \in \mathbb{R}$ and $\delta > 0$ and define as in the lecture the Cerami-neighbourhoods

$$N^C_{\beta,\delta} := \{u \in X \mid |E(u) - \beta| < \delta,\, \|dE(u)\|_{X^*} < \frac{\delta}{1 + \|u\|_X}\}.$$

We want to find a flow similar to the pseudo-gradient flow in Lemma 2.2.1. from the lecture. Therefore proceed as follows:

(a) Find a vectorfield $\tilde{e}^C \colon \tilde{X} \to X$, where $\tilde{X} = \{u \in X \mid dE(u) \neq 0\}$, satisfying for all $u \in \tilde{X}$:

$$\begin{aligned} \|\tilde{e}^C(u)\|_X &< 1 + \|u\|_X \\ \langle dE(u), \tilde{e}^C(u)\rangle &> \frac{1}{2}\|dE(u)\|_{X^*}\Big(1 + \|u\|_X\Big). \end{aligned}$$

(b) Assume $N^C_{\beta,\delta} = \emptyset$. Define a vector field $e \colon X \to X$ by multiplying $\tilde{e}^C$ with an appropriate cut-off function. Let $\Phi \colon X \times \mathbb{R} \to X$ be defined by

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(u,t) = e(\Phi(u,t)) \\ \Phi(u,0) = u. \end{cases}$$

Show that the trajectories of Φ exist for all times.

(c) Complete the proof by finding $\varepsilon > 0$ such that $\Phi(E_{\beta+\varepsilon}, 1) \subseteq E_{\beta-\varepsilon}$ and by proving that $t \mapsto E(\Phi(u,t))$ is non-increasing.