Problem Set 9

1. Präsenzaufgabe. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth, bounded domain, where $n \geq 3$. Consider

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The corresponding functional is

$$\begin{aligned} E \colon H_0^1(\Omega) &\to \mathbb{R} \cup \{\pm\infty\} \\ u &\mapsto \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \int_\Omega G(x, u(x))\, dx, \end{aligned}$$

where $G(x,s) = \int_0^s g(x,t)\, dt$ is the primitive of g. Assume the following conditions:

1. $g(x, 0) = 0$ and $\limsup\limits_{s\to 0} \frac{g(x,s)}{s} \leq 0$ uniformly in $x \in \Omega$.
2. There is $p < 2^*$ and a constant C such that $|g(x,s)| \leq C(1 + |s|^{p-1})$ for almost every $x \in \Omega$, $s \in \mathbb{R}$.
3. There is $q > 2$ and a radius R_0 such that $0 < qG(x,s) \leq g(x,s)s$ for almost every $x \in \Omega$, if $|s| \geq R_0$.

Show that $E \in C^1(H_0^1(\Omega))$ and that E satisfies $(\text{P.-S.})_\beta$ for all $\beta \in \mathbb{R}$.

2. Boundary Value Problem. Consider the setting from Exercise 1. Show that one can apply the mountain pass Lemma to the functional E and use it to prove that there are non-trivial solutions $u^+ \geq 0 \geq u^-$ of the boundary value problem (1).

Hint: Consider the function

$$g^+(x,u) = \begin{cases} g(x,u) & u > 0 \\ 0 & u \leq 0 \end{cases}$$

to find a solution u^+ and a similar function to find u^-.

3. Three Distinct Solutions. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain and let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ be the eigenvalues of $-\Delta$ on $H_0^1(\Omega) \cap H^2(\Omega)$. Consider for $f \in L^2(\Omega)$ the problem

$$\begin{cases} -\Delta u = \lambda u - u^3 + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that for each $\lambda \in (\lambda_1, \lambda_2)$ there is a constant $C(\lambda)$ such that for $\|f\|_2 < C(\lambda)$, there are at least three distinct solutions of this problem. Proceed as follows:

(a) Consider the functional

$$
\begin{aligned}
E\colon H_0^1(\Omega) &\to \mathbb{R} \\
u &\mapsto \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx + \frac{1}{4}\int_\Omega |u|^4\, dx - \int_\Omega uf\, dx.
\end{aligned}
$$

Show that E attains minima $u^\pm \in M^\pm$, where

$$
\begin{aligned}
M^+ &:= \{u \in H_0^1(\Omega) \mid (u, e_1)_{H_0^1} > 0\}, \\
M^- &:= \{u \in H_0^1(\Omega) \mid (-u, e_1)_{H_0^1} > 0\},
\end{aligned}
$$

where e_1 is the first eigenfunction of $-\Delta$ and where the scalar product is

$$
(u, v)_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v\, dx.
$$

(b) Use Theorem 2.3.2. from the lecture to find a third critical value.