Solution 1

1. The Gateaux derivative of E can be computed as follows:

$$\begin{aligned} DE(u)v &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(u+\varepsilon v) \\ &= \int_\Omega \nabla u \cdot \nabla v \pm |u|^{p-2}uv + fv\,dx. \end{aligned}$$

We need to show $DE(u)$ is a bounded, linear (i.e. continuous) functional on $L^p \cap H^1(\Omega)$ for each u. Therefore we use Hölder three times to estimate:

$$\begin{aligned} DE(u)v &= \int_\Omega \nabla u \cdot \nabla v \pm |u|^{p-2}uv + fv\,dx \\ &\le \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |u|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |v|^p\,dx\right)^{1/p} + \dots \\ &\qquad \dots + \left(\int_\Omega f^2\right)^{\frac{1}{2}} \left(\int_\Omega v^2\right)^{\frac{1}{2}} \\ &= \|\nabla u\|_2 \|\nabla v\|_2 + \|u\|_p^{p-1}\|v\|_p + \|f\|_2\|v\|_2 \\ &\le C(u)(\|v\|_{H^1} + \|v\|_p). \end{aligned}$$

$\|v\|_{H^1} + \|v\|_p$ is one of several equivalent norms on $L^p(\Omega) \cap H^1(\Omega)$, so $\|DE(u)\|_{\mathrm{op}} \le C(u)$.

Finally we need to show that the map $L^p(\Omega) \cap H^1(\Omega) \to (L^p(\Omega) \cap H^1(\Omega))^*$, $u \mapsto DE(u)$ is continuous. Therefore, let $u_0 \in L^p(\Omega) \cap H^1(\Omega)$ arbitrary and u close enough to u_0 (will be determined later). Then:

$$\begin{aligned} \|DE(u) - DE(u_0)\|_{\mathrm{op}} &= \sup_{\|v\|=1} \Big| DE(u)v - DE(u_0)v \Big| \\ &= \sup \Big| \int_\Omega (\nabla u - \nabla u_0)\cdot \nabla v \pm (|u|^{p-2}u - |u_0|^{p-2}u)v\,dx \Big| \\ &\le \sup \|\nabla u - \nabla u_0\|_2 \|\nabla v\|_2 + \int_\Omega (|u|^{p-2}u - |u_0|^{p-2}u)v\,dx, \end{aligned}$$

where we used Hölder as above. (The f-term vanishes.) The first term converges to 0, if $u \to u_0$ in H^1. For the second term, we use the following Theorem:

Theorem. *Let $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. If the non-linear operator*

$$\begin{aligned} T\colon L^p(\Omega) &\to L^q(\Omega) \\ u &\mapsto g(\cdot, u(\cdot)) \end{aligned}$$

is well-defined, then it is also continuous.

The proof of this Theorem (and the definition of Carathéodory function) will be given below in an Appendix. One can solve the last part as well using Example 3.4.2. from M.Struwe: Analysis III / Mass und Integral, which is almost the same statement.

To use the theorem we realize that $T\colon L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega)$, $u \mapsto |u|^{p-2}u$ is well-defined, where $g(x,u) = |u|^{p-2}u$. So the Theorem states that if $\|u-u_0\|_p \to 0$, then $\||u|^{p-2}u-|u_0|^{p-2}u\|_{\frac{p}{p-1}} \to 0$ and therefore

$$\left(\int_\Omega (|u|^{p-2}u - |u_0|^{p-2}u)v\,dx\right)^p \leq \left\||u|^{p-2}u - |u_0|^{p-2}u\right\|^{p-1}_{\frac{p}{p-1}} \|v\|_p \to 0, \text{ as } u \to u_0.$$

This implies $E \in C^1$, which then also shows that the Gateaux-derivative is in fact the Fréchet-derivative.

2. Recall the Sobolev-embedding: For $n = 2$ we have $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$, for all $1 < q < \infty$. For $n > 2$ we have $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ for $1 < q \leq \frac{2n}{n-2}$; $\frac{2n}{n-2}$ is the Sobolev exponent.

Using Exercise 1 we therefore have that $E \in C^1(H^1_0(\Omega))$ if either $n = 2$ and $1 < p < \infty$ of if $n > 2$ and $1 < p \leq \frac{2n}{n-2}$.

If $n > 2$ and $p > \frac{2n}{n-2}$, there exist functions in $H^1_0(\Omega)$ not lying in $L^p(\Omega)$. For such a function u, $DE(u)$ is not well defined as can be seen when inserting $v = u$.

3. **(a)** The Euler-Lagrange equation is given by $DE(u) = 0$. In this case we have

$$\begin{aligned} DE(u)v &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega f(x, u+\varepsilon v, \nabla u + \varepsilon \nabla v)\,dx \\ &= \int_\Omega \frac{\partial f}{\partial q}(x,u,\nabla u)v + \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i}(x,u,\nabla u)\right)\frac{\partial v}{\partial x_i}\,dx \\ &= \int_\Omega \frac{\partial f}{\partial q}(x,u,\nabla u)v - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial p_i}(x,u,\nabla u)\right) v\,dx. \end{aligned}$$

This has to vanish for all $v \in H^1_0$, so we have by the fundamental lemma of calculus of variations and as $u \in C^2$:

$$\frac{\partial f}{\partial q}(x,u,\nabla u) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial p_i}(x,u,\nabla u)\right) = 0.$$

(b) (This is one possible solution, there are other conditions, which are sufficient.)

$$\begin{aligned} &\operatorname{div}(a(x,u(x),\nabla u(x))) + b(x,u(x),\nabla u(x)) = 0 \\ \Rightarrow &\int_\Omega \Big(-\operatorname{div}(a(x,u(x),\nabla u(x))) + b(x,u(x),\nabla u(x))\Big)v\,dx = 0 \text{ for all } v \\ \Rightarrow &\int_\Omega a(x,u(x),\nabla u(x)) \cdot \nabla v + b(x,u(x),\nabla u(x))v\,dx = 0 \end{aligned}$$

Comparing with the calculations in (a) we have to find f satisfying:

$$\begin{aligned} &\frac{\partial f}{\partial q}(x,u,\nabla u) = b(x,u,\nabla u) \\ &\frac{\partial f}{\partial p_i}(x,u,\nabla u) = a_i(x,u,\nabla u) \text{ for all } i, \end{aligned}$$

where a_i means the i-th component of a. It exists a function f satisfying these if we have:

$$\frac{\partial b}{\partial p_i} = \frac{\partial a_i}{\partial q} \text{ for all } i \text{, and}$$
$$\frac{\partial a_i}{\partial p_j} = \frac{\partial a_j}{\partial p_i} \text{ for all } i \text{ and } j.$$

Appendix: Proof of the Theorem We prove a slightly generalized version, the proof follows M. Krasnosel'skii: Topological Methods in the Theory of Nonlinear Integral Equations.

Definition 1 (Carathéodory function)**.** Let U and V be topological spaces and $(\Omega, \mathcal{A}, dx)$ a measure space. $f\colon \Omega \times U \to V$ is called *Carathéodory function* if

(i) $f(\cdot, u)\colon \Omega \to V$ is measurable for each $u \in U$,
(ii) $f(x, \cdot)\colon U \to V$ is continuous for each $x \in \Omega$.

Theorem 2. *Let $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. If the non-linear operator*

$$\begin{aligned} T\colon L^p(\Omega) &\to L^q(\Omega) \\ u &\mapsto g(\cdot, u(\cdot)) \end{aligned}$$

is well-defined, then it is also continuous.

For the proof we use the following Lemma:

Lemma 3 (Nemytskii)**.** *Let $|\Omega| < \infty$. Then the operator T preserves convergence in measure.*

Proof. Let $\{u_n\}_n \subset L^p(\Omega)$ be a sequence converging in measure to $u \in L^p(\Omega)$.

Fix $\varepsilon > 0$ and define the subsets

$$G_n^{(k)} = \{x \in \Omega \mid |u(x) - u_n(x)| < \tfrac{1}{k} \Rightarrow |g(x, u(x)) - g(x, u_n(x))| < \varepsilon\}.$$

Clearly $G_n^{(k)} \subset G_n^{(k+1)}$ for all k. We view x as a parameter and appeal to the continuity of $g(x, \cdot)$ for any $x \in \Omega$ to obtain

$$\bigcup_{k \in \mathbb{N}} G_n^{(k)} = \Omega.$$

(Otherwise, $\xi \in \Omega$ exists with $|u(\xi) - u_n(\xi)| < \frac{1}{k}$ for any k but $|g(\xi, u(\xi)) - g(\xi, u_n(\xi))| \geq \varepsilon$ which contradticts the continuity of $g(\xi, \cdot)$)

Fix $\eta > 0$ and choose $k_0 \in \mathbb{N}$ such that $G_n := G_n^{(k_0)}$ satisfies

$$|\Omega \setminus G_n| < \tfrac{\eta}{2}.$$

Now, we consider the subsets

$$U_n = \{x \in \Omega \mid |u(x) - u_n(x)| < \tfrac{1}{k_0}\},$$
$$D_n = \{x \in \Omega \mid |g(x, u(x)) - g(x, u_n(x))| < \varepsilon\}.$$

Convergence $u_n \to u$ in measure implies that there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|\Omega \setminus U_n| < \tfrac{\eta}{2}.$$

From $U_n \cap G_n \subset D_n$ we conclude

$$|\Omega \setminus D_n| \leq |\Omega \setminus (U_n \cap G_n)| = |(\Omega \setminus U_n) \cup (\Omega \setminus G_n)| \leq |\Omega \setminus U_n| + |\Omega \setminus G_n| < \eta$$

for all $n \geq N$ which implies convergence $Tu_n \to Tu$ in measure as $\varepsilon, \eta > 0$ were arbitrary. $\square$

Proof of the Theorem. We prove the Theorem for spaces of finite measure $|\Omega| < \infty$. First we show continuity at 0. If T is not continuous in $0 \in L^p(\Omega)$, then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ and some $a > 0$ such that

$$\|\varphi_n\|_p \xrightarrow{n\to\infty} 0, \qquad \|T\varphi_n\|_q > a^{\frac{1}{q}} \quad \forall n \in \mathbb{N}.$$

We construct numbers $\varepsilon_k > 0$, sets $G_k \subset \Omega$ and a subsequence $\{n_k\}_k$ such that

(a) $$\varepsilon_{k+1} < \tfrac{1}{2}\varepsilon_k,$$
(b) $$|G_k| \leq \varepsilon_k,$$
(c) $$\int_{G_k} |T\varphi_{n_k}|^q \, dx > \tfrac{2}{3}a,$$
(d) $$\forall D \subset \Omega,\ |D| < 2\varepsilon_{k+1} : \int_D |T\varphi_{n_k}|^q \, dx < \tfrac{1}{3}a$$

inductively. Let the induction start at $\varepsilon_1 = |\Omega|$, $G_1 = \Omega$ and $n_1 = 1$. Suppose (b) and (c) hold up to $k \in \mathbb{N}$ with ε_k, n_k, G_k already known. There exists $\varepsilon_{k+1} > 0$ such that (d) holds. This is due to $T\varphi_{n_k} \in L^q(\Omega)$. The number ε_{k+1} automatically satisfies (a) since φ_{n_k} satisfies (c). Depending on ε_{k+1}, there exist $n_{k+1} \in \mathbb{N}$ and $G_{k+1} \subset \Omega$ such that

$$|T\varphi_{n_{k+1}}|^q \leq \frac{a}{3|\Omega|} \quad \text{in } \Omega \setminus G_{k+1} \text{ and } |G_{k+1}| < \varepsilon_{k+1}.$$

This follows as $T\varphi_n$ converges in measure to zero according to the Lemma. Therefore, (b) holds also for $k+1$. It remains to verify, that $\varphi_{n_{k+1}}$ and G_{k+1} satisfy (c). Indeed,

$$\int_{G_{k+1}} |T\varphi_{n_{k+1}}|^q \, dx = \int_\Omega |T\varphi_{n_{k+1}}|^q \, dx - \int_{\Omega \setminus G_{k+1}} |T\varphi_{n_{k+1}}|^q \, dx > a - \tfrac{1}{3}a = \tfrac{2}{3}a.$$

Consider the disjoint subsets

$$D_k = G_k \setminus \bigcup_{j=k+1}^{\infty} G_j$$

and observe that by conditions (a) and (b)

$$|G_k \setminus D_k| = \Big|\bigcup_{j=k+1}^{\infty} G_j\Big| \le \sum_{j=k+1}^{\infty} \varepsilon_j < 2\varepsilon_{k+1}.$$

Let $\psi\colon \Omega \to \mathbb{R}$ be given by the following concatenation

$$\psi(x) = \begin{cases} \varphi_{n_k}(x), & \text{if } k \in \mathbb{N} \text{ with } x \in D_k \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

When choosing $\varphi_n \to 0$, we may assume $\sum_{n=1}\|\varphi_n\|_p < \infty$ or switch to a subsequence with this property. Therefore, clearly $\psi \in L^p(\Omega)$. Since T is well-defined, $T\psi \in L^q(\Omega)$. However, for any any $k \in \mathbb{N}$

$$\int_{D_k} |T\psi|^q \, dx = \int_{D_k} |T\varphi_{n_k}|^q \, dx \ge \int_{G_k} |T\varphi_{n_k}|^q \, dx - \int_{G_k \setminus D_k} |T\varphi_{n_k}|^q \, dx > \tfrac{2}{3}a - \tfrac{1}{3}a = \tfrac{1}{3}a,$$

as $|G_k \setminus D_k| < 2\varepsilon_{k+1}$. Recalling that the subsets D_k are disjoint, a contradiction to $T\psi \in L^q$ arises through

$$\int_{\Omega} |T\psi|^q \, dx \ge \sum_{k=1}^{\infty} \int_{D_k} |T\psi|^q \, dx = \infty.$$

Consequently, $T\colon L^p \to L^q$ cannot be well-defined at ψ, if it is not continuous in 0. Let us now deduce continuity of T at any $u_0 \in L^p(\Omega)$.

$$\tilde{g}(x,u) = g(x, u_0 + u) - g(x, u_0)$$

is a Carathéodory function inducing a well-defined Operator $\tilde{T}\colon L^p \to L^q$ with $T0 = 0$. As shown above, $\tilde{T}$ is continuous in 0. But this implies that $T\colon u \mapsto g(\cdot, u)$ is continuous in u_0.

$\square$