Solution 10

1. Präsenzaufgabe: Pseudo-Gradient Vector Field.

(a) The idea is as follows: The map $d\pi(u)\colon H^1(\mathbb{S}^1,\mathbb{R}^3) \to T_uM$ is surjective, so we can find any $w = w(u) \in H^1(\mathbb{S}^1,\mathbb{R}^3)$ such that $d\pi(u)w \in T_uM$ satisfies the desired conditions. Then for all $v \in M$, the element $d\pi(v)w \in T_vM$ is defined. We claim that there is some neighbourhood U_u of u such that $d\pi(v)w$ satisfies the desired properties for all $v \in U_u$. If this is proven we then proceed as usual by taking a locally finite refinement of the cover $\{U_u\}$ and a Lipschitz continuous partition of unity to sum up the corresponding vectors.

To prove the claim, recall that

$$\|(d\pi(u) - d\pi(v))w\|_{H^1} \le C\|\nu(u) - \nu(v)\|_{H^1}\|w\|_{H^1},$$

where both $d\pi(u)w$ and $d\pi(v)w$ are viewed as elements of H^1. As the normal vector field ν is smooth, and both u and v are (Hölder) continuous by Sobolev embedding, we can make this difference arbitrarily small and therefore $\|d\pi(v)w\| < 1$ for v close to u. From smoothness it also follows that $d\pi(v)w$ is Lipschitz in a neighbourhood around u. As $E \in C^1(M)$ also $dE(u)$ depends continuously on u and therefore $\langle dE(v), d\pi(v)w\rangle_{T_v^*M\times T_vM} > \frac{1}{2}\|dE(v)\|$ will as well be satisfied for v close enough to u.

(b) The above construction nowhere used that $S \subseteq \mathbb{R}^3$, so it works analogue in the case of a hypersurface in $\mathbb{R}^n$. If the codimension $\operatorname{codim}(S,\mathbb{R}^n) > 1$, then we no longer have a normal vector field but instead a normal vector bundle (whose dimension is $k = \operatorname{codim}(S,\mathbb{R}^n)$). Then we choose an ONB $e_1(x),\ldots,e_k(x)$ of the normal vector bundle, depending continuously on $x \in S$ and then $d\pi(x)v = v - (e_1(x)\cdot v)e_1(x) - \ldots - (e_k(x)\cdot v)e_k(x)$. The proof works then similarly.

2. Flow Invariant Family. Let

$$\tilde{P} = \{p \in C^0([0,\pi]^{n-2}; H^1(\mathbb{S}^1,S))\}.$$
$$P = \{p \in \tilde{P} \mid p(\vartheta_1,\ldots,\vartheta_{n-2}) \text{ is a constant path if it exists } i : \vartheta_i \in \{0,\pi\}\}.$$

Each $p \in P$ then induces a map $p\colon \mathbb{S}^{n-1} \to S$ using the parametrization of $\mathbb{S}^{n-1}$ via $S^1 \times [0,\pi]^{n-2}$. Call the diffeomorphism $\Psi\colon S \to \mathbb{S}^{n-1}$. Then we define the familiy

$$\mathcal{F} = \{p \in P \mid \Psi(p(\vartheta_1,\ldots,\vartheta_{n-2})(\varphi)) \text{ is homotopic to } id\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}\}.$$

Let now $\Phi(t)$ be a 1-parameter family of homeomorphisms of $H^1(\mathbb{S}^1,S)$. Note that $\Phi(0) = id$. Let $p \in \mathcal{F}$, i.e. the map $\Psi(p(\vartheta_1,\ldots,\vartheta_{n-2})(\varphi))$ is homotopic to the identity.

$\Phi(1)\circ p\colon [0,\pi]^{n-2} \to H^1(\mathbb{S}^1,S)$ is homotopic to p via the homotopy $\Phi(t)\circ p$. (Note that all $\Phi(t)\circ p$ are in particular elements of P, because $\Phi(t)$ maps constant maps to constant maps.) Then we can use this homotopy in the sense of

$$(\varphi,\vartheta_1,\ldots,\vartheta_{n-2},t) \mapsto \Psi(\Phi(t)\circ p(\vartheta_1,\ldots,\vartheta_{n-2})(\varphi))$$

to show that $\Psi(p(\vartheta_1,\dots,\vartheta_{n-2})(\varphi))$ is homotopic to $\Psi(\Phi(1)\circ p(\vartheta_1,\dots,\vartheta_{n-2})(\varphi))$ which implies that the latter is homotopic to the identity, so $\mathcal{F}$ is Φ-invariant.

3. Closed Geodesic. With exactly the same steps as in the lecture we can show that E satisfies $(\text{P.-S.})_\beta$ for all $\beta\in\mathbb{R}$. As the family $\mathcal{F}$ is flowinvariant, we know that

$$\beta = \inf_{p\in\mathcal{F}} \sup_{\vartheta_i\in[0,\pi]} E(p(\vartheta_1,\dots\vartheta_{n-2}))$$

is a critical value. We now need to show two things: If $dE(u)=0$ for some u, then u is a closed geodesic and on the other hand that $\beta>0$ so that the closed geodesic we obtain is not a trivial one.

Assume $dE(u)=0$. Then for every $\varphi\in H^1(\mathbb{S}^1,\mathbb{R}^n)$:

$$\begin{aligned}0&=\langle dE(u),\varphi-\nu(u)(\nu(u)\cdot\varphi)\rangle\\&=\int_0^1 \dot u\dot\varphi-\dot u\tfrac{d}{ds}\Big(\nu(u)(\nu(u)\cdot\varphi)\Big)\,ds\\&=\int \dot u\dot\varphi-\dot u\cdot\tfrac{d}{ds}(\nu(u))(\nu(u)\cdot\varphi)\,ds,\end{aligned}$$

because $\dot u(x)$ lies in the tangent space $T_{u(s)}S$ and therefore is orthogonal to $\nu(u(s))$. So we get that in the sense of distributions

$$\ddot u = -\Big(\dot u\cdot d\nu(u)\dot u\Big)\nu(u). \tag{1}$$

$\dot u\cdot d\nu(u)\dot u$ is an L^1-function whereas $\nu(u)$ is bounded. So $u\in W^{2,1}\hookrightarrow C^1$. Then $\dot u\in C^0$ and again by the above formula $u\in C^2$ and in particular it satisfies (1) as function (not just in the distributional sense), i.e. $\ddot u$ is proportional to $\nu(u)$ or, in other words, orthogonal to the tangent space. This is the definition of a geodesic.

To prove that $\beta>0$ we first bound the diameter of a function $u\in H^1(\mathbb{S}^n,\mathbb{R}^n)$ by

$$|u(t)-u(s)|\le\int_s^t|u(\dot r)|\,dr\le\sqrt{|t-s|}(2E(u))^{\frac12},\quad 0\le t,s\le 1.$$

This is $<d$ (the radius in which the nearest neighbour projection is defined) if $E(u)<\frac{d^2}{2}=:\beta_0$. We will show that $\beta\ge\beta_0$.

Assume not. Then there is $p\in\mathcal{F}$ satisfying $\sup_{\vartheta_1,\dots,\vartheta_{n-2}\in[0,\pi]} E(p(\vartheta_1,\dots\vartheta_{n-2}))<\beta_0$. We will prove that p is homotopic to a constant map. This gives then a contradiction, as $\Phi\circ p$ was assumed to be homotopic to the identiy and the identiy map $\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$ is not homotopic to a trivial map.

Consider the following homotopy:

$$\begin{aligned}h(\vartheta_1,\dots,\vartheta_{n-2},s)(\varphi)&=\pi\Big(p(\vartheta_1,\dots,\vartheta_{n-2})(0)+s(p(\vartheta_1,\dots,\vartheta_{n-2})(\varphi)-p(\vartheta_1,\dots,\vartheta_{n-2})(0))\Big)\\&=\pi\Big(sp(\vartheta_1,\dots,\vartheta_{n-2})(\varphi)+(1-s)p(\vartheta_1,\dots,\vartheta_{n-2})(0)\Big).\end{aligned}$$

From the first form we see that this is indeed well defined because $|p(t') - p(t)| < d$ for all choices of choices of t and t' as caluculated above. From the second form we see that this is a homotopy between $p(\vartheta_1, \ldots, \vartheta_{n-2})(\varphi)$ and the constant maps $p(\vartheta_1, \ldots, \vartheta_{n-2})(0)$. We can now find a further homotopiy contracting this to a constant map:

$$\tilde{h}(\vartheta_1, \ldots \vartheta_{n-2}, s_1)(\varphi) = p(s_1\vartheta_1 \ldots \vartheta_{n-2})(0)$$

which contracts everything to the map $\tilde{p}(\vartheta_1, \ldots, \vartheta_{n-2})(\varphi) = p(0, \vartheta_1, \ldots, \vartheta_{n-2})(\varphi)$, which is a constant map by definition of $\mathcal{F}$.