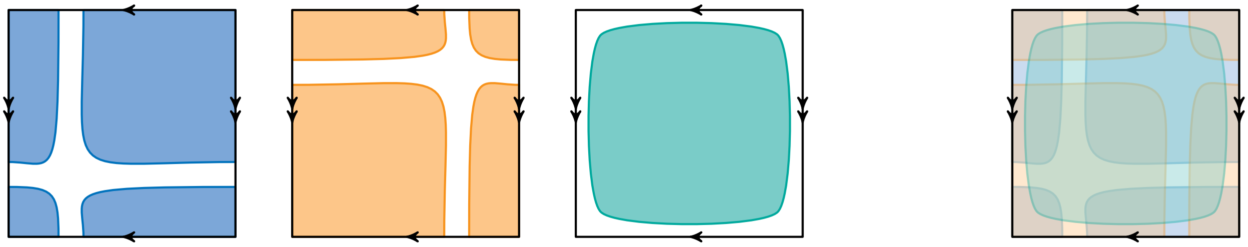
Solution 11

1. Category.

(a) $\mathbb{S}^2$ is not contractible, therefore $\mathrm{cat}_{\mathbb{S}^2}(\mathbb{S}^2) > 1$. On the other hand we can cover $\mathbb{S}^2$ by two contractible open sets, for example $\mathbb{S}^2$ without a small open disc at the northpole and $\mathbb{S}^2$ without an open disc at the southpole.

(b) We can cover $\mathbb{T}^2$ be three contractible sets, for example as follows:



Assume we can cover $\mathbb{T}^2$ be just two contractible sets. This would imply that $\mathbb{T}^2 \setminus \{p\}$ is contractible, where p is the point to which we contract one of the two sets. But it is known that this space has nontrivial fundamental group, so we get $\mathrm{cat}_{\mathbb{T}^2}(\mathbb{T}^2) > 2$.

(c) We can cover $\mathbb{R}\mathrm{P}^2$ by three contractible sets, for example by taking as first set the sphere without a small belt along the equator. Under the identification $x \sim -x$, this set becomes a contractible (and in particular connected) subset of $\mathbb{R}\mathrm{P}^2$. For the two other sets, we take the sphere without small belts along great circles orthogonal to the equator and to each other. These sets are all contractible and they cover $\mathbb{R}\mathrm{P}^2$ as long as the belts are small enough.

Assume, by contradiction, there is a covering of $\mathbb{R}\mathrm{P}^2$ with two contractible sets U_1 and U_2. As $\mathbb{R}\mathrm{P}^2$ has no boundary we can assume (by perturbing the two sets a bit if necessary) that there is an ε-neigbourhood of the boundary of U_1 which is completely covered by U_2 for some $\varepsilon > 0$ small enough. We now use relative homology to derive a contradiction. Therefore we consider the long exact sequence of the pair $(\mathbb{R}\mathrm{P}^2, U_1)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(U_1) \arrow[r] & H_2(\mathbb{R}\mathrm{P}^2) \arrow[r] & H_2(\mathbb{R}\mathrm{P}^2, U_1) \arrow[dll] & \\
 & H_1(U_1) \arrow[r] & H_1(\mathbb{R}\mathrm{P}^2) \arrow[r] & H_1(\mathbb{R}\mathrm{P}^2, U_1) \arrow[dll] & \\
 & H_0(U_1) \arrow[r] & H_0(\mathbb{R}\mathrm{P}^2) \arrow[r] & H_0(\mathbb{R}\mathrm{P}^2, U_1) \arrow[r] & 0
\end{tikzcd}$$

U_1 is contractible, so its homology is $H_0(U_1) = \mathbb{Z}$ and 0 else. To compute $H_i(\mathbb{R}\mathrm{P}^2, U_1)$ we use excision. Therefore we take away $Z = U_1 \setminus N_\varepsilon(\partial U_1)$, where $N_\varepsilon(\partial U_1)$ is the ε-neigbourhood of its boundary. Because then, $\mathbb{R}\mathrm{P}^2 \setminus Z$ is homotopy equivalent to U_2 which is homotopy equivalent to a disc and $U_2 \setminus Z$ is homotopy equivalent to the boundary of the disc. Therefore

$H_2(\mathbb{R}\mathrm{P}^2, U_1) = H_2(D^2, \partial D^2) = \mathbb{Z}$ and 0 else, where D^2 is the 2-dimensional disc. Inserting this into the diagram we get

$$\begin{array}{l} \cdots \longrightarrow 0 \longrightarrow H_2(\mathbb{R}\mathrm{P}^2) \longrightarrow \mathbb{Z} \\ \longrightarrow 0 \longrightarrow H_1(\mathbb{R}\mathrm{P}^2) \longrightarrow 0 \\ \longrightarrow \mathbb{Z} \longrightarrow H_0(\mathbb{R}\mathrm{P}^2) \longrightarrow 0 \longrightarrow 0 \end{array}$$

It is known that the homology if $\mathbb{R}\mathrm{P}^2$ is

$$H_i(\mathbb{R}\mathrm{P}^2) = \begin{cases} 0 & i = 2 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}.$$

This contradicts the above long exact sequence which would imply $H_1 = 0$.

2.

(a) We check the condition for an index. Let $A, A_1, A_2 \in \mathcal{A}$ and $h\colon M \to M$ a homeomorphism.

i) $\mathrm{cat}_M(A) \geq 0$ follows from the definition. $\mathrm{cat}_M(A) = 0$ implies that A can be covered by no set, so A needs to be the empty set.

ii) Assume $A_1 \subseteq A_2$. Then a cover of A_2 is in particular a cover of A_1.

iii) Let $U_1, \dots, U_k$ be a minimal contractible cover for A_1 and $V_1, \dots, V_l$ a minimal cover for A_2 (w.l.o.g. finite covers exist, else the statement is trivial). Then $U_1, \dots, U_k, V_1, \dots, V_l$ is a cover for $A_1 \cup A_2$ i.e. $\mathrm{cat}_M(A_1 \cup A_2) \leq \mathrm{cat}_M(A_1) + \mathrm{cat}_M(A_2)$.

iv) The preimage of a contractible set under the homeomorphism h is contractible. Because assume V is a contractible set and $F\colon [0,1] \times V \to M$ is the homotopy between the inclusion $V \hookrightarrow M$ and the map sending all of V to some point $x_0 \in M$. Then $\tilde{F}\colon h^{-1}(V) \to X$, $\tilde{F}(t,x) = h^{-1}(F(t, h(x)))$ is well defined and a homotopy between the inclusion $h^{-1}(V) \hookrightarrow X$ and the map sending $h^{-1}(V)$ to $h^{-1}(x_0) \in M$. Let $V_1, \dots, V_k$ be a minimal cover with contractible sets of $h(A)$ (if no such cover exists, the statement is trivial), then $h^{-1}(V_1), \dots, h^{-1}(V_k)$ is a cover with contractible sets of A.

v) Assume A is compact. Let $\{U_i\}_{i \in I}$ be any covering with open sets contractible in M such that their closures $\overline{U}_i$ are still contractible in M. Then we can find a finite subcover $U_1, \dots, U_\ell$ still covering A, hence $\mathrm{cat}_M(A) \leq \ell < \infty$.

   Let $k = \mathrm{cat}_M(A)$ and $V_1, \dots V_k$ be a minimal contractible cover. For each V_i there is a homotopy $F : V_i \times [0,1] \to M$, with $F_0(x) = x$, $F_1(x) = p_0$, for some $p_0 \in M$. We extend F to $(V_i \times [0,1]) \cup (M \times \{0\}) \cup (M \times \{1\})$ by letting F be the identity on $M \times \{0\}$

and constant p_0 on $M \times \{1\}$. We then embed M into some $\mathbb{R}^N$ for N large enough. Then $F : V_i \times [0,1] \to \mathbb{R}^N$ and we can use Tietze componentwise to extend F to a neighbourhood U of $(V_i \times [0,1]) \cup (M \times \{0\}) \cup (M \times \{1\})$. Then we can find an open W_i with $V_i \subseteq W_i \subseteq M$ and $W_i \times [0,1] \subseteq U$. Then W_i and its closure W_i are contractible by using the extended map F and we let $N = W_1 \cup \ldots \cup W_k$ be the open neighbourhood of A. By construction, $\mathrm{cat}_M(N) = \mathrm{cat}_M(A)$.

vi) Assume $A = \{p_1, \ldots, p_n\}$ is finite. Then we can show by induction that there is a contractible set containing all these points. Assume we have a closed set K_{n-1} which is contractible and contains $\{p_1, \ldots, p_{n-1}\}$. If $p_n \in K_{n-1}$, we are done, else take a path $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p_n$, $\gamma([0,1)) \cap K_{n-1} = \emptyset$ and $\gamma(1) \in K_{n-1}$. (This path can be found by connecting p_n with any point in K_{n-1} and stopping as soon as we hit K_{n-1}.) Set $K_n = K_{n-1} \cup \gamma([0,1])$. K_n is contractible, because we can first find a continuous map $M \to M$ mapping $K_n \to K_{n-1}$ (moving along γ) and K_{n-1} is contractible by assumption. By construction, K_n is closed.

(b) Note first that f satisfies (P.-S.)$_\beta$ for all $\beta \in \mathbb{R}$, because M is assumed to be compact. We proceed in a similar way as in Theorem 2.5.5. for the Krasnoselskii genus. For $1 \le k \le \mathrm{cat}_M(M)$

$$\beta_k = \inf_{A \in \mathcal{A}, \mathrm{cat}_M(A) \ge k} \sup_{u \in A} f(u).$$

We first claim that each β_k is a critical value. Indeed, assume $K_{\beta_k} = \emptyset$ for some k. Then let $0 \le \varepsilon \le \bar{\varepsilon} = 1$ and $\Phi \in C^0(M \times [0,1]; M)$ be the flow as constructed in Theorem 2.3.3. Let further $A \in \mathcal{A}$ with $\mathrm{cat}_M(A) \ge k$ be such that $\sup_{u \in A} f(u) < \beta_k + \varepsilon$. As $\Phi(\cdot, 1)$ is a homeomorphism, $\Phi(A, 1) =: A_1 \in \mathcal{A}$. By property iv) it holds $\mathrm{cat}_M(A_1) \ge \mathrm{cat}_M(A) \ge k$, i.e. A_1 is a comparison set for β_k. Furthermore, $A_1 = \Phi(A, 1) \subseteq \Phi(E_{\beta+\varepsilon}, 1) \subseteq E_{\beta-\varepsilon}$, which is a contradiction.

To prove there are at least $\mathrm{cat}_M(M)$ critical points, we need to consider what happens if $\beta_k = \beta_j =: \beta$ for some $k > j$. We claim that in such a case, $\mathrm{cat}_M(K_\beta) > k - j \ge 1$. By vi) this implies that K_β is not finite. Let therefore such $k > j$ be given. As K_β is compact, by condition v) we can find $N \supseteq K_\beta$ open with $\mathrm{cat}_M(\overline{N}) = \mathrm{cat}_M(K_\beta)$. Let $0 < \varepsilon < \bar{\varepsilon} = 1$ and Φ be as constructed in Theorem 2.3.3. for N. Take $A \in \mathcal{A}$ with $\mathrm{cat}_M(A) \ge k$ and $\sup_{u \in A} f(u) < \beta + \varepsilon$ and let as before $A_1 = \Phi(A, 1) \in \mathcal{A}$ with $\mathrm{cat}_M(A_1) \ge k$. Then $A_1 \subseteq \Phi(E_{\beta+\varepsilon}, 1) \subseteq \overline{E_{\beta-\varepsilon}} \cup \overline{N}$.

Note that $\mathrm{cat}_M(\overline{E_{\beta-\varepsilon}}) < j$, because else it would be a comparison set for $\beta_j = \beta$. Then by ii) and iii)

$$\begin{aligned} k &\le \mathrm{cat}_M(A_1) \le \mathrm{cat}_M(\overline{E_{\beta-\varepsilon}} \cup \overline{N}) \\ &\le \mathrm{cat}_M(\overline{E_{\beta-\varepsilon}}) + \mathrm{cat}_M(\overline{N}) < j + \mathrm{cat}_M(\overline{N}). \end{aligned}$$

This implies $\mathrm{cat}_M(K_\beta) = \mathrm{cat}_M(\overline{N}) > k - j$, which proves the claim and ends the proof.

3. Billard To find a sensible map $\varphi\colon D \to \mathbb{R}\mathrm{P}^2$, we note that $f(0,t) = f(1,t) = f(t,1)$, so it makes sense to identify the two points $(0,t), (t,1) \in D$. Furtermore, $f \equiv 0$ on all of the diagonal $s = t$ and also at the point $(0, 1)$. Therefore we construct φ by first sending D to the disc $\mathbb{D}^2$, where we map all points (s, s) to the same point in the boundary and $(0, 1)$ is mapped to the antipodal point. We then continuously extend this map such that $(0, t)$ and $(t, 1)$ are mapped to antipodal points on the boundary. $\mathbb{D}^2$ can then be mapped to $\mathbb{R}\mathrm{P}^2$ by identifying antipodal points on the boundary.

The construction of φ suggests, how $\tilde{f}$ needs to look like, because by construction, for all points $p \in \mathbb{R}\mathrm{P}^2$, the preimage $\varphi^{-1}(p)$ is mapped under f to the same value. Therefore $\tilde{f}$ is induced by f.

Note that the smoothness of $\tilde{f}$ is at least as good as the one of f, so $\tilde{f} \in C^\infty(\mathbb{R}\mathrm{P}^2)$. Then by Exercise 2, $\tilde{f}$ has at least $\mathrm{cat}_{\mathbb{R}\mathrm{P}^2}(\mathbb{R}\mathrm{P}^2)$ critical points and by Exercise 1(c), this number is three.