Solution 12

1. Perturbation Theory.

(a) Fix the notation as follows:

$$\begin{aligned}
\beta_j &= \inf_{h\in\Gamma}\sup_{u\in X_j} E(h(u)),\\
\delta_j &= \inf_{h\in\Gamma}\sup_{u\in X_j} \tilde{E}(h(u)),\\
X_{j+1}^+ &= \{v + t\varphi_{j+1} \mid v\in X_j,\, t\geq 0\},\\
\delta_j^* &= \inf_{h\in\Gamma}\sup_{u\in X_{j+1}^+} \tilde{E}(h(u)),\\
X &= H_0^1(\Omega).
\end{aligned}$$

We want to apply Theorem 2.6.2. from the lecture. $\tilde{E}$ satisfies (P.-S.)$_\beta$ for every $\beta\in\mathbb{R}$ and for $W = X_j$, $w^* = \varphi_{j+1}$ conditions i) to iii) from the Theorem are satisfied. So the only thing left to show is $\delta_j^* > \delta_j$, then there is a critical value $\gamma_j^*\geq\delta_j^*$, which will be $>\beta_j$, as we will see in the calculations below. The calculations will mostly follow the proof of Theorem 2.6.3.

Claim 1. For all $\varepsilon>0$ and every j, there is a constant $B = B(\varepsilon, j)$ such that $|\beta_j - \delta_j| < \varepsilon$, whenever $\|f\|_2 < B$.

Proof. Choose $h\in\Gamma$ such that $\sup_{u\in X_j} E(h(u)) < \beta_j + \frac{\varepsilon}{2}$.

We can find a radius $R>0$ such that for all $u\in X_j$ with $\|u\|_{H_0^1} > R$ it holds $\tilde{E}(u)\leq 0$ uniformly for all f with $\|f\|_2 < 1$. As X_j is finite dimensional, there is a constant L such that $h(\overline{B_R^{X_j}}(0))\subseteq B_L^X(0)$, again for all f with $\|f\|_2<1$. Then we can estimate:

$$\begin{aligned}
\delta_j &\leq \sup_{u\in X_j}\tilde{E}(h(u)) = \sup_{u\in X_j,\tilde{E}(u)\geq 0}\tilde{E}(h(u))\\
&\leq \sup_{u\in X_j,\tilde{E}(u)\geq 0} E(h(u)) + \|f\|_2\|\underbrace{h(u)}_{\in B_L(0)}\|_2\\
&\leq \beta_j + \frac{\varepsilon}{2} + \|f\|_2 L\sqrt{\lambda_1},
\end{aligned}$$

where $\lambda_1 = \inf_{0\neq v\in C_c^\infty(\Omega)}\dfrac{\|\nabla u\|_2^2}{\|u\|_2^2}$. When choosing $B_1(\varepsilon,j) = \frac{\varepsilon}{2L\sqrt{\lambda_1}}$ we get $\delta_j - \beta_j < \varepsilon$ and starting with $\tilde{E}$ we get $\beta_j - \delta_j < \varepsilon$ for all $\|f\|_2 < B_2(\varepsilon,j)$ and then we just need to take $B = \min\{B_1, B_2\}$. □

This claim first of all implies that for $\|f\|_2 < B(\frac{\beta_{j+1}-\beta_j}{2}, j)$ it holds

$$\delta_j < \frac{\beta_j + \beta_{j+1}}{2}.$$

Now we want to show $\delta_j^* > \frac{\beta_j+\beta_{j+1}}{2}$ for $\|f\|_2$ small enough, then we are done.

Assume $h \in \Gamma$ is such that $\sup_{u\in X_{j+1}^+} \tilde{E}(h(u)) < \delta_j^* + \xi$, where $\xi > 0$ is a number which will be determined later.

Claim 2. We may assume that $\sup_{u\in X_{j+1}^+} \tilde{E}(h(u))$ is attained in

$$N := \{u \in H_0^1(\Omega) \mid \|d\tilde{E}(u)\|_{H^{-1}} + \|d\tilde{E}(-u)\|_{H^{-1}} \leq 12\|f\|_{H^{-1}}\}$$

in the sense of $\sup_{u\in X_{j+1}^+} \tilde{E}(h(u)) = \sup_{v\in N} \tilde{E}(v)$.

Proof. Let $\tilde{e}$ be the p.g.v.f. for E as in Theorem 2.5.1. and set

$$e(v) = -\tau(\max\{\tilde{E}(v), \tilde{E}(-v)\})\eta(v)\tilde{e}(v),$$

with $\tau \in C^\infty(\mathbb{R})$, $\tau(s) = 1$ if $s \geq \xi$, $\tau \in [0,1]$ and $\tau(s) = 0$ if $s \leq 0$. η even with $\eta(v) = 1$ for $v \notin N$, $\eta(v) = 0$ for $\|d\tilde{E}(v)\|_{H^{-1}} + \|d\tilde{E}(-v)\|_{H^{-1}} \leq 8\|f\|_{H^{-1}}$.

Then similar to the lecture, for all $v \notin N$ with $\max\{\tilde{E}(v), \tilde{E}(-v)\} \geq \xi$ we have

$$\begin{aligned}
-\langle d\tilde{E}(v), e(v)\rangle &= \langle dE(v) - f, \tilde{e}(v)\rangle \\
&\geq \frac{1}{2}\|dE(v)\| - \|f\| \\
&\geq \frac{1}{4}\Big(\|d\tilde{E}(v)\| + \|d\tilde{E}(-v)\| - 8\|f\|\Big) \\
&\geq \|f\|.
\end{aligned}$$

So taking large enough t, we will get that $\Phi(\cdot, t) \circ h$ satisfies the claim, because else we would fall below level δ_j^*. □

We can then estimate:

$$\begin{aligned}
\delta_j^* &\geq \sup_{u\in X_{j+1}^+} \tilde{E}(h(u)) - \xi \\
&\geq \sup_{u\in X_{j+1}} \tilde{E}(h(u)) - \sup_{\substack{v\in N,\\ \min\{\tilde{E}(v),\tilde{E}(-v)\}\leq\delta_{j+1}+\xi}} |\tilde{E}(v) - \tilde{E}(-v)| - \xi \\
&\geq \delta_{j+1} - \xi - 2\|f\|_2 \sup_{\substack{v\in N,\\ \min\{\tilde{E}(v),\tilde{E}(-v)\}\leq\delta_{j+1}+\xi}} \|v\|_2.
\end{aligned}$$

Claim 3. For all $v \in N$ with $\min\{\tilde{E}(u), \tilde{E}(-u)\} \leq \delta_{j+1} + \xi$ there is a constant $A = A(\xi, j)$ such that

$$\|u\|_2 \leq A$$

uniformly for all $\|f\|_2 < B(1, j)$.

Proof. We can assume w.l.o.g. $\tilde{E}(u) \leq \delta_{j+1} + \xi$, because $\|-u\| = \|u\|$.

On one hand we have

$$\begin{aligned} pE(u) - \langle dE(u), u\rangle &= \frac{p}{2}\|\nabla u\|_2^2 - \|p\|_p^1 - \|\nabla u\|_2^2 + \|u\|_p^2 \\ &= \frac{p-2}{2}\|\nabla u\|_2^2 = c_0\|u\|_{H_0^1}^2. \end{aligned}$$

On the other hand

$$\begin{aligned} pE(u) - \langle dE(u), u\rangle &= p\tilde{E}(u) + p\int fu\,dx - \langle d\tilde{E}(u), u\rangle - \langle f, u\rangle \\ &\leq p\tilde{E}(u) - \langle d\tilde{E}(u), u\rangle + c_1\|u\|_2\|f\|_2 \\ &\leq p(\delta_{j+1} + \xi) + 12\|f\|_2\|u\| + c_1\|u\|_2 B \\ &\leq p(\beta_{j+1} + 1 + \xi) + 12\|u\|_2 B + c_1\|u\|_2 B \\ &\leq c_2 + c_3\|u\|_{H_0^1}, \end{aligned}$$

where c_2 depends on j and ξ. These two inequalities combined we get

$$c_0\|u\|_{H_0^1}^2 \leq c_2 + c_3\|u\|_{H_0^1},$$

from which we get a bound on $\|u\|_{H_0^1}$, which then gives a bound for $\|u\|_2$. □

Inserting this, we get:

$$\begin{aligned} \delta_j^* &\geq \delta_{j+1} - \xi - 2\|f\|_2 A \\ &> \frac{\beta_{j+1} + \beta_j}{2}, \end{aligned}$$

if we let $\xi = \frac{\beta_{j+1} - \beta_j}{8}$ and then let C_j, the bound on $\|f\|_2$ be the smaller of the numbers $B(\frac{\beta_{j+1}-\beta_j}{8}, j+1)$ as in Claim 1 or the number needed to have $2\|f\|_2 A < \frac{\beta_{j+1}-\beta_j}{8}$.

(b) Let $\rho > 0$, $h \in \Gamma$. Call $Y_j = \overline{\operatorname{span}\{\varphi_j, \varphi_{j+1}, \dots\}}$ and $S_\rho = \{u \in H_0^1(\Omega) \mid \|u\|_{H_0^1} = \rho\}$ as in the lecture. Lemma 2.6.1. tells us that $\gamma(h(X_j) \cap S_\rho \cap Y_j) \geq 1$, so in particular $h(X_j) \cap S_\rho \cap Y_j \neq \emptyset$. This implies that for every $h \in \Gamma$, the image of X_j under h has to intersect with $S_\rho \cap Y_j$, so it holds

$$\sup_{u \in X_j} E(h(u)) \geq \inf_{u \in Y_j, \|u\| = \rho} E(u)$$

for all $h \in \Gamma$, so also

$$\begin{aligned} &\beta_j \geq \inf_{u \in Y_j, \|u\| = \rho} E(u) \\ \Rightarrow\ &\beta_j \geq \sup_{\rho > 0} \inf_{u \in Y_j, \|u\| = \rho} E(u). \end{aligned}$$

To calculate $E(u) = \frac{1}{2}\|u\|^2_{H^1_0} - \frac{1}{p}\|u\|^p_p$, we use the interpolation formula with $r = n(1 - \frac{p}{2^*}) = p - \frac{n(p-2)}{2}$ and get

$$\begin{aligned} &\|u\|^p_p \leq \|u\|^r_2 \|u\|^{p-r}_{2^*} \\ \Rightarrow\ &E(u) \geq \frac{1}{2}\rho^2 - C\lambda_j^{-\frac{r}{2}}\rho^p, \end{aligned}$$

where we used $\|u\|_2 \leq \lambda_j^{-\frac{1}{2}}\|u\|_{H^1_0}$ for all $u \in Y_j$ and $\|u\|_{2^*} \leq S^{-1}\|u\|_{H^1_0}$, where S is the Sobolev constant.

To compute the supremum over all ρ we differentiate w.r.t. ρ and get that the supremum is attained at $\rho = \tilde{C}\lambda_j^{\frac{r}{2(p-2)}}$. Inserting this we get

$$\beta_j \geq c_0 \lambda_j^{\frac{r}{p-2}}.$$

Note that the exponent $\frac{r}{p-2}$ is positive for $2 < p < 2^*$, so this tends to ∞ for $j \to \infty$.

To get more precise estimates, we can use Weyl's law, which tells that $N(x) \sim x^{\frac{n}{2}}$, where $N(x)$ is the number of eigenvalues (with multiplicity) smaller than x, i.e. $N(x) = |\{j \mid \lambda_j \leq x\}|$, from which follows $\lambda_j \sim j^{\frac{2}{n}}$. Inserting this:

$$\beta_j \geq c_1 j^{\frac{2r}{n(p-2)}} = c_1 j^{\frac{2p}{n(p-2)}-1} \to \infty.$$

(c) Choose $J = J(k) \in \mathbb{N}$ such that at least k of the γ_j^* (see part (a)) are different for $j = 1, 2, \ldots, J$. Then set $\tilde{C}_k = C_J$.