Solution 2

1. Präsenzaufgabe

(a) Let $u \in M$. To show coerciveness of E, we estimate $E(u)$ from below.

$$E(u) = \int_0^d (\sqrt{1+|u'|^2} + u)\,dx \geq \int_0^d \sqrt{1+|u'|}\,dx - \int_0^d |u|\,dx.$$

To estimate the negative term we use the following trick.

$$x \in [0, \tfrac{d}{2}] \;\Rightarrow\; |u(x)| = \Big|\int_0^x u'(y)\,dy\Big| \leq \int_0^{\frac{d}{2}} |u'(y)|\,dy \leq \int_0^{\frac{d}{2}} \sqrt{1+|u'|}\,dy,$$
$$x \in [\tfrac{d}{2}, d] \;\Rightarrow\; |u(x)| = \Big|-\int_x^d u'(y)\,dy + u(d)\Big| \leq \int_{\frac{d}{2}}^d |u'(y)|\,dy + 1 \leq \int_{\frac{d}{2}}^d \sqrt{1+|u'|}\,dy + 1.$$

This allows us to estimate

$$\begin{aligned}\int_0^d |u(x)|\,dx &= \int_0^{\frac{d}{2}} |u(x)|\,dx + \int_{\frac{d}{2}}^d |u(x)|\,dx \\ &\leq \frac{d}{2}\int_0^{\frac{d}{2}} \sqrt{1+|u'|^2}\,dy + \frac{d}{2}\int_{\frac{d}{2}}^d \sqrt{1+|u'|^2}\,dy + \frac{d}{2} \\ &= \frac{d}{2}\int_0^d \sqrt{1+|u'|^2}\,dy + \frac{d}{2}.\end{aligned}$$

Therefore,

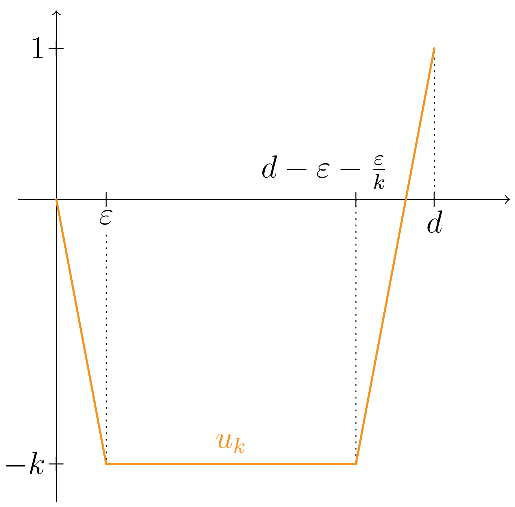
$$E(u) \geq (1-\frac{d}{2})\int_0^d \sqrt{1+|u'|^2}\,dx - \frac{d}{2}.$$

As $(1-\frac{d}{2}) > 0$, we get that E is coercive as claimed.

(b) Consider the sequence $(u_k) \subset M$ where $u_k(x)$ is given by the values $u_k(0) = 0$, $u_k(\varepsilon) = -k$, $u_k(d-\varepsilon-\frac{\varepsilon}{k}) = -k$, $u_k(d) = 1$ and linear interpolation in between. We assume $0 < \varepsilon \ll 1 \leq k$ and compute

$$\begin{aligned}\int_0^d u_k\,dx &= -k(d-\varepsilon-\tfrac{\varepsilon}{k}) + \tfrac{\varepsilon}{2k} \\ &< -k(d-\varepsilon)+1,\end{aligned}$$
$$\begin{aligned}\int_0^d \sqrt{1+|u_k'|^2}\,dx &= (2\varepsilon+\tfrac{\varepsilon}{k})\sqrt{1+(\tfrac{k}{\varepsilon})^2} \\ &\leq (2\varepsilon+\tfrac{\varepsilon}{k})(\tfrac{k}{\varepsilon}+\tfrac{\varepsilon}{2k}) \\ &= 2k+1+\tfrac{\varepsilon^2}{k}+\tfrac{\varepsilon^2}{2k^2} \\ &\leq 2k+2,\end{aligned}$$

where we used $\sqrt{1+s^2} \leq (s+\frac{1}{2s})$ for $s > 0$.



We conclude

$$E(u_k) \leq 2k - k(d-\varepsilon) + 3 = (2-d+\varepsilon)k + 3.$$

If $d > 2$, there exists $\varepsilon > 0$ such that $(2-d+\varepsilon) < 0$ and we obtain $E(u_k) \to -\infty$ for $k \to \infty$.

(c) For $d = 2$ we choose the sequence u_k from above and set $\varepsilon = \frac{1}{k}$ for each $k \in \mathbb{N}$. Then the same estimate yields $E(u_k) \leq \varepsilon k + 3 = 4$, which contradicts coerciveness as $\|u_k\|_{W^{1,1}} \to \infty$. From the final estimate in (a) we deduce $E(u) \geq -1$ if $d = 2$.

2. Minimisation of General Functionals.

We want to apply Theorem 1.1.1. from the lecture to this functional. Therefore we realize that $H^1_0(\Omega)$ is reflexive and E is coercive because $E(u) = \int_\Omega f(x,u,\nabla u)\,dx \geq \int_\Omega |\nabla u|^2\,dx$, which tends to ∞ whenever $\|u\|_{H^1_0} \to \infty$.

What is left to show is that E is weakly sequentially lower semicontiuous. Let $u_m \overset{\mathrm{w}}{\rightharpoonup} u$ in $H^1_0(\Omega)$. By considering only the subsequence on which $(E(u_k))_k$ approaches its limes inferior we may assume that $(E(u_k))_k$ is convergent.

Since the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we may extract a subsequence such that $u_k \to u$ converges in $L^2(\Omega)$ which again allows a subsequence converging pointwise almost everywhere. We will use this to conclude Step 2.

Step 1. As $\nabla u_k \overset{\mathrm{w}}{\rightharpoonup} \nabla u$ in $L^2(\Omega)$, by Mazur's Lemma (Satz 4.6.2. from the FAI-script), there exist norm-convergent convex linear combinations

$$P_\ell = \sum_{m=m_0}^{\ell} a^m_\ell \nabla u_m \xrightarrow{\|\cdot\|_{L^2(\Omega)}} \nabla u, \qquad \sum_{m=m_0}^{\ell} a^m_\ell = 1, \qquad a^m_\ell > 0.$$

We may reduce $(P_\ell)_\ell$ to a subsequence converging pointwise almost everywhere. This makes $(f(x,u,P_\ell))_\ell$ a sequence of non-negative functions converging pointwise on Ω neglecting a set of measure zero and the Lemma of Fatou applies.

$$\begin{aligned}
\int_\Omega f(x,u,\nabla u)\,dx &\leq \liminf_{\ell\to\infty} \int_\Omega f(x,u,P_\ell)\,dx && \text{(Fatou)}\\
&\leq \liminf_{\ell\to\infty} \sum_{m=m_0}^{\ell} a^m_\ell \int_\Omega f(x,u,\nabla u_m)\,dx && \text{(convexity)}\\
&\leq \sup_{m\geq m_0} \int_\Omega f(x,u,\nabla u_m)\,dx\\
&\leq \limsup_{m\to\infty} \int_\Omega f(x,u,\nabla u_m)\,dx. && (m_0 \to \infty)
\end{aligned}$$

Step 2. (Eisen) The functions $g_m = f(\cdot,u_m,\nabla u_m) - f(\cdot,u,\nabla u_m)$ converge in measure to zero.

To ease notation, the prefix $\{x \in \Omega \mid \ldots\}$ is omitted when specifying sets. Assume, for contradiction, there exists $\varepsilon_0 > 0$ such that for every m in a subsequence $\Lambda_1 \subset \mathbb{N}$

$$0 < \delta \leq \Big|\Big\{|g_m(x)| \geq \varepsilon_0\Big\}\Big|.$$

The weakly convergent sequence ∇u_m is bounded in $L^2(\Omega)$ uniformly with respect to m (Banach-Steinhaus). Therefore, there exists some large b (of order $\delta^{-1/2}$) such that

$$\Big|\Big\{|\nabla u_m(x)| \geq b\Big\}\Big| < \tfrac{1}{2}\delta$$

for all m, which then implies $\frac{1}{2}\delta < |\Omega_m|$, where

$$\Omega_m := \Big\{|g_m(x)| \geq \varepsilon_0,\ |\nabla u_m(x)| < b\Big\}.$$

Moreover, since Ω is bounded, the set Q of all $x \in \Omega$ which appear for infinitely many m has positive measure $|Q| > 0$. Note that

$$Q := \bigcap_{n=1}^{\infty} \bigcup_{\Lambda_1 \ni m \geq n} \Omega_m.$$

It therefore intersects nontrivially with $W = \{u_m(x) \to u(x)\}$, the set of pointwise convergence. Choose $x_0 \in Q \cap W$ and collect $\Lambda_2 = \{m \in \Lambda_1 \mid x_0 \in \Omega_m\}$. By construction, $\nabla u_m(x_0)$ is bounded and therefore converges on a subsequence $\Lambda_3 \subset \Lambda_2$ to some $p \in \mathbb{R}^n$. Since convexity in $\mathbb{R}^n$ implies continuity, we may conclude both

$$\begin{aligned} f\Big(x_0, u(x_0), \nabla u_m(x_0)\Big) &\to f(x_0, u(x_0), p), \\ f\Big(x_0, u_m(x_0), \nabla u_m(x_0)\Big) &\to f(x_0, u(x_0), p) \end{aligned}$$

as $\Lambda_3 \ni m \to \infty$. This finally contradicts $|g_m(x_0)| \geq \varepsilon_0$.

Step 3. Since g_m converges in measure, we can extract a subsequence which converges pointwise almost everywhere. Moreover, by Egorov's Theorem, for every $\delta > 0$ there exists a set Ω_δ of measure $|\Omega_\delta| < \delta$ such that $g_m(x)$ converges uniformly with respect to $x \in \Omega \setminus \Omega_\delta$. In particular, for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $m \geq N$ and every $x \in \Omega \setminus \Omega_\delta$

$$f\Big(x, u(x), \nabla u_m(x)\Big) < f\Big(x, u_m(x), \nabla u_m(x)\Big) + \varepsilon.$$

Fix $\varepsilon > 0$. Applying the integral estimates from Step 1 in $\Omega \setminus \Omega_\delta$ we obtain

$$\begin{aligned} \int_{\Omega\setminus\Omega_\delta} f(x, u, \nabla u)\, dx &\leq \limsup_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} f(x, u, \nabla u_m)\, dx \\ &\leq \limsup_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} f(x, u_m, \nabla u_m)\, dx + \varepsilon|\Omega| \\ &\leq \lim_{m\to\infty} E(u_m) + \varepsilon|\Omega|. \qquad (f \geq 0) \end{aligned}$$

Finally, for any $\eta > 0$ there exists $\delta > 0$ and a corresponding set Ω_δ such that

$$\int_{\Omega_\delta} f(x, u, \nabla u)\, dx \leq \eta$$

since $f(x, u, \nabla u) \leq C|\nabla u|^2 + C$ is integrable. We conclude

$$E(u) \leq \lim_{m\to\infty} E(u_k) + \varepsilon|\Omega| + \eta$$

for any $\varepsilon, \eta > 0$ which completes the proof.

3. Weak Maximum Principle

(a) Consider the function $\varphi(x) = u_+(x) = \max\{u(x), 0\}$. We have that $\varphi \in H_0^1(\Omega)$ with weak derivative

$$\nabla\varphi(x) = \begin{cases} \nabla u(x) & \text{if } u \geq 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

Inserting φ into the test equation we get:

$$\begin{aligned} 0 &\geq \sum \int_\Omega a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u_+}{\partial x_j} \, dx + \int_\Omega c u u_+ \, dx \\ &= \sum \int_\Omega a^{ij} \frac{\partial u_+}{\partial x_i} \frac{\partial u_+}{\partial x_j} \, dx + \int_\Omega c u u_+ \, dx \\ &\geq \lambda \|\nabla u_+\|^2 + \int_\Omega c u u_+ \, dx \geq \lambda \|\nabla u_+\|^2 \end{aligned}$$

where the second-last inequality follows from the uniform positiv definiteness of (a^{ij}) and the last inequality follows from positiveness of c. Therefore $\nabla u_+ = 0$ a.e. and as $u_+ \in H_0^1$, $u_+ = 0$ a.e., so $u \leq 0$ on Ω.

(b) The same calculations as above can be applied to $u + \alpha$, if $\alpha \leq 0$ and the conclusion will hold similarly. We can use this for $\alpha = \inf_{\partial\Omega} u \leq 0$. Then we can as well modify the proof so that for u a weak supersolution with $u \geq 0$ on $\partial\Omega$ we have $u \geq 0$ a.e.