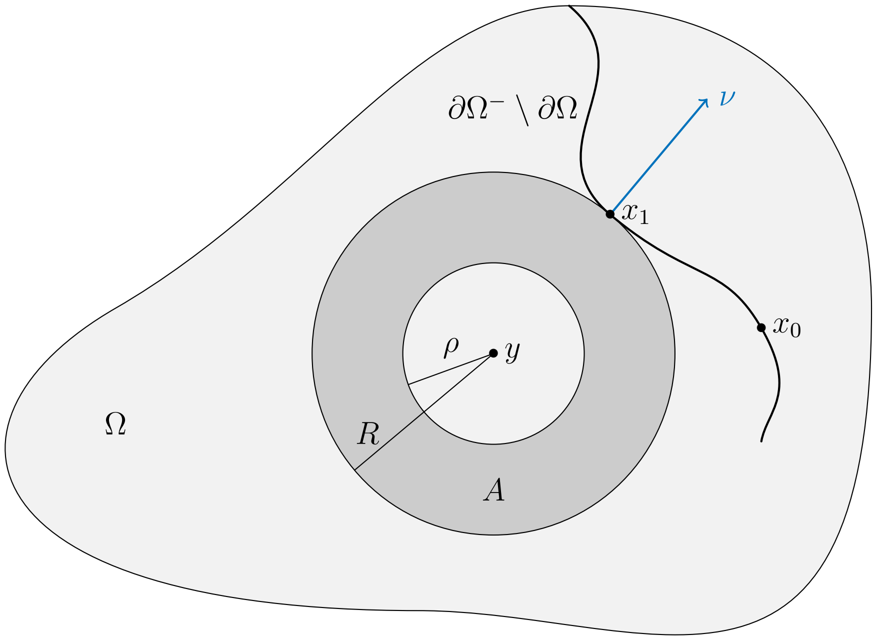
Solution 3

1. Strong Maximum Principle Assume u is nonconstant and $u(x_0) = 0$. Let Ω^- be the set of all x, s.t. $u(x) < 0$. Let $B_R(y) \subseteq \Omega^-$ be a ball touching $\partial\Omega^- \setminus \partial\Omega$, which exists as u is nonconstant and continuous. Then for some point x_1 at the boundary of $B_R(y)$ it holds $u(x_1) = 0$. We will show that $\frac{\partial u}{\partial \nu}(x_1) > 0$, where ν denotes the outward unit normal of $B_R(y)$ at x_1. This is Eberhard[1] Hopf's Lemma. But this is a contradiction to the fact that u attains its maximum at x_1. (By the weak maximum principle, the maximum is 0).



Let $0 < \rho < R$ and define the function $v(x) = e^{-\alpha r(x)^2} - e^{-\alpha R^2}$ on the annulus $A := B_R(y) \setminus B_\rho(y)$, where $r(x) = |x - y| > \rho$. α is a constant, which will be determined later.

Let, for simplicity of calculations, in the following $Lu = -\Delta u + cu$. Then we get:

$$\begin{aligned}
Lv &= -\sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} v + cv \\
&= -\sum_i \frac{\partial}{\partial x_i}(-2\alpha(x_i - y_i)e^{-\alpha r^2}) + cv \\
&= \sum_i 2\alpha e^{-\alpha r^2} - 4\alpha^2 (x_i - y_i)^2 e^{-\alpha r^2} + cv \\
&= 2n\alpha e^{-\alpha r^2} - 4\alpha^2 r^2 e^{-\alpha r^2} + c(e^{-\alpha r^2} - e^{-\alpha R^2}).
\end{aligned}$$

As $r > \rho > 0$, we can choose α big enough s.t. $Lv \le 0$ for all $x \in A$.

For $\varepsilon > 0$ small enough, we have $u - u(x_1) + \varepsilon v \le 0$ on ∂A. This holds, because $u(x) < u(x_1)$ for $x \in \partial B_\rho(y)$ and $v(x) = 0$ for $x \in \partial B_R(y)$. Using $Lv \le 0$ and linearity of L we have

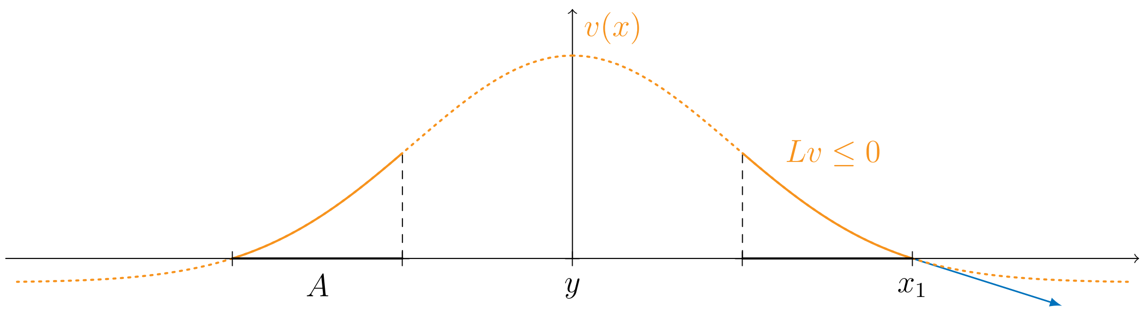
[1] not to confuse with our Heinz Hopf

$L(u - u(x_1) + \varepsilon v) \leq -cu(x_1) \leq 0$ in the weak sense. Therefore, using the weak maximum principle, we get $u - u(x_1) + \varepsilon v \leq 0$ on all of A.This implies

$$\frac{u(x_1) - u(x)}{|x_1 - x|} \geq \frac{\varepsilon v(x) - \varepsilon v(x_1)}{|x_1 - x|},$$

as $v(x_1) = 0$. Taking the correct limit we therefore get $\frac{\partial u}{\partial \nu}(x_1) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_1) > 0$.

The following is an illustration of the function v:



2.

(a) We can always write $c = c_+ - c_-$. When we then define

$$\begin{aligned}\tilde{L}u &= -\sum_{i,j} \frac{\partial}{\partial x_i} a^{ij} \frac{\partial u}{\partial x_j} + c_+ u \\ &= Lu + c_- u\end{aligned}$$

we see that $\tilde{L}u \leq 0$, because $Lu \leq 0$ and $c_- u \leq 0$. (Here we need the assumption that $u \leq 0$, taking any other constant would not be enough.) Then we can apply Exercise 1 to this problem with the operator $\tilde{L}$ and $M = 0$ to get that $u \equiv 0$ a.e.

(b) No, this cannot hold in general, as one sees from the eigenfunctions of $-\Delta$. There is $u \in H_0^1(\Omega)$ satisfying $-\Delta u = \lambda u$ in Ω, but u is not trivial, i.e. $|u| > 0$ somewhere in Ω.

3. Let $g(x) = u_1(x) + u_2(x)$, which is continuous by continuity of u_1 and u_2. (g comes from taking $\frac{u_2^2 - u_1^2}{u_2 - u_1}$, where possible, and extending continuously.) We want to apply the maximum principle from Exercise 2 to the operator $L = -\Delta - g(x)$ and the function $u_2 - u_1$. We immediately see that $u_2 - u_1 = 0$ on $\partial\Omega$ as both u_1 and u_2 solve the boundary value problem. By the assmuptions it also follows $u_2 - u_1 \leq 0$ everywhere on Ω. Lastly we have $L(u_2 - u_1) = 0$, so by Exercise 2, either $u_2 - u_1 \equiv 0$ everywhere or $u_2 - u_1$ is nowhere 0.

4. There are several ways to prove this, two are presented here:

Version 1: Assume $\frac{\partial u}{\partial \nu} = 0$ on all of $\partial\Omega$. Then we can extend u to all of $\mathbb{R}^n$ be setting it 0 outside Ω and this extended u will be a function in $H_0^1(\mathbb{R}^n)$. This function is a solution of

$$\begin{cases} -\Delta u = u|u|^{p-2} & \text{in } \mathbb{R}^n, \\ \quad u \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Restricting this to $U \subseteq \mathbb{R}^n$ bounded with $\bar{\Omega} \subset U$ we get that by the strong maximum principle $u \equiv 0$, because for $x \in U \setminus \Omega$ we have $u(x) = 0$.

Version 2: Eberhard Hopf's Lemma tells that if $u = 0$ on the boundary and $u > 0$ in the interior, then $\frac{\partial u}{\partial \nu} < 0$ on the boundary. Which is a contradiction to $\frac{\partial u}{\partial \nu} = 0$. Eberhard Hopf's Lemma was proven implicitely in Exercise 1.