Solution 4

1. Präsenzaufgabe As a_∞ is a constant and therefore in particular radial, we can apply Remark 2 from the lecture to E_∞ to find $\bar{u} \in M$ radial, i.e. $\bar{u}(x) = \bar{u}(|x|)$, satisfying

$$E_\infty(\bar{u}) = \inf_{u \in M} E_\infty(u) = I_\infty.$$

As $a < a_\infty$ we have for all $u \in M$: $E(u) < E_\infty(u)$. This holds in particular for $\bar{u}$, therefore

$$I \leq E(\bar{u}) < E_\infty(\bar{u}) = I_\infty.$$

2.

(a) Let $u \in M_\lambda$. Then $u \in L^2 \cap L^{2^*}(\mathbb{R}^n)$ and interpolation yields $u \in L^p(\mathbb{R}^n)$ for $2 < p < 2^*$ as well as the estimate

$$\|u\|_p \leq \|u\|_2^\gamma \|u\|_{2^*}^{1-\gamma},$$

where $\gamma \in (0,1)$ is given by $\frac{1}{p} = \frac{\gamma}{2} + \frac{1-\gamma}{2^*}$. Applying Sobolev's embedding we get

$$\|u\|_p \leq C\|u\|_2^\gamma \|\nabla u\|_2^{1-\gamma}.$$

Therefore, using that $\|u\|_2 = \lambda^{\frac{1}{2}}$:

$$\|u\|_p^p \leq C\|\nabla u\|_2^{p(1-\gamma)}.$$

Solving for $\gamma = \frac{2n-2p(n-2)}{np-p(n-2)}$ we obtain $p(1-\gamma) = \frac{np-2n}{2} < 2$, when assuming $p < \frac{2n+4}{n}$.

As $a(x)$ converges for $|x| \to \infty$ we can bound $|a(x)| \leq A$ globally and therefore

$$\begin{aligned} E(u) &\geq \|\nabla u\|_2^2 - A\|u\|_p^p \\ &\geq \|\nabla u\|_2^2 - CA\|\nabla u\|_2^{p(1-\gamma)} \end{aligned}$$

and as $p(1-\gamma)$ is strictly less than 2 we have $E(u) \to \infty$ if $\|\nabla u\|_2 \to \infty$. ($\|u\|_{H^1} \to \infty$ implies $\|\nabla u\|_2 \to \infty$ as $\|u\|_2$ is constant on M_λ.)

(b) Let $\varepsilon > 0$. Recalling that $C_c^\infty(\mathbb{R}^n) \subseteq H^1(\mathbb{R}^n)$ is dense, we can choose $u_1, u_2 \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned} I_{\lambda-\alpha} &\leq \quad E(u_1) \leq I_{\lambda-\alpha} + \varepsilon, \quad \|u_1\|_2^2 = \lambda - \alpha; \\ I_{\alpha,\infty} &\leq E_\infty(u_2) \leq I_{\alpha,\infty} + \varepsilon, \quad \|u_2\|_2^2 = \alpha. \end{aligned}$$

Taking $x_0 \in \mathbb{R}^n$ with $|x_0|$ large enough we have that the supports of $u_1(x)$ and $u_{2,x_0}(x) = u_2(x - x_0)$ are disjoint. By choosing $|x_0|$ even larger if needed, we may additionaly assume

$$\int_{\mathbb{R}^n} |a(x) - a_\infty||u_{2,x_0}|^p \, dx \leq \varepsilon,$$

i.e. $|E(u_{2,x_0}) - E_\infty(u_{2,x_0})| \le \varepsilon$. Then we get

$$\|u_1 + u_{2,x_0}\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2 = \lambda - \alpha + \alpha = \lambda,$$
$$I_\lambda \le E(u_1 + u_{2,x_0}) = E(u_1) + E(u_{2,x_0}) \le I_{\alpha,\infty} + I_{\lambda-\alpha} + 3\varepsilon.$$

As ε was arbitrary, we get the desired inequality.

(c) Assume equality, $I_\lambda = I_{\alpha,\infty} + I_{\lambda-\alpha}$. Then choose minimizing sequences $u_k \in C_c^\infty(\mathbb{R}^n)$ for E_∞ on M_α, and $v_k \in C_c^\infty(\mathbb{R}^n)$ for E on $M_{\lambda-\alpha}$.

As in (b), choose for each k some $x_k \in \mathbb{R}^n$ with $|x_k|$ large enough such that the supports of u_{k,x_k} and v_k are disjoint and such that $|E(u_{k,x_k}) - E_\infty(u_{k,x_k})| \le \frac{1}{k}$. Then we get

$$\begin{aligned} I_\lambda &\le E(u_{k,x_k} + v_k) \\ &= E(u_{k,x_k}) + E(v_k) \\ &\le E_\infty(u_{k,x_k}) + \tfrac{1}{k} + E(v_k) \\ &\le I_{\alpha,\infty} + I_{\lambda-\alpha} + o(1) = I_\lambda + o(1), \quad k \to \infty. \end{aligned}$$

Therefore, $(u_{k,x_k} + v_k)$ is a minimizing sequence for E on M_λ and if we assume $|x_k| \to \infty$, this sequence has no convergent subsequence in M_λ.

(d) Assume w.l.o.g $\lambda = 1$. (For general λ, we need to normalize the functions to use the Concentrations Compactness Lemma, but this will not change the statement.) Let u_k be a minimzing sequence for E on M_1. By definition of M_1 we have that $\mu_k = u_k^2\, dx$ are probability measures. The Concentration-Compactness-Lemma I then states that there is a subsequence, still denoted μ_k, such that either i) Compactness or ii) Vanishing or iii) Dichotomy occurs. We will rule out cases ii) and iii) and show that in case i) there is a convergent subsequence of the u_k.

ii) Assume Vanishing. Given $\varepsilon > 0$, let R be a radius s.t. $|a(x) - a_\infty| \le \varepsilon$ for all $|x| > R$. By coercivity shown in (a) we have that the minimizing sequence $(u_k)_k$ is bounded in $H^1(\mathbb{R}^n)$. Using $\|u_k\|_p \le C\|u_k\|_2^\gamma \|\nabla u_k\|_2^{1-\gamma}$ we get:

$$\begin{aligned} E(u_k) - E_\infty(u_k) &= \int_{\mathbb{R}^n} (a(x) - a_\infty)|u_k|^p\, dx \\ &\le \varepsilon \|u_k\|_{L^p(\mathbb{R}^n)}^p + 2A\|u_k\|_{L^p(B_R)}^p \\ &\le \varepsilon C' + C'' \|u_k\|_{L^2(B_R)}^{p\gamma} \|\nabla u_k\|_{L^2(B_R)}^{p(1-\gamma)}, \end{aligned}$$

where A is the bound on $|a(x)|$. By the vanishing property we have $\|u_k\|_{L^2(B_R)} \to 0$ for $k \to \infty$. Hence, for $\varepsilon \to 0$, $I_{1,\infty} \le I_1$, which contradicts $I_1 < I_{\alpha,\infty} + I_{1-\alpha}$ for $\alpha = 1$.

iii) Assume Dichotomy with parameter $\alpha \in (0,1)$. Recall from the proof of the Concentration-Compactness-Lemma that the two dichotomous measures come from restricting the original measures, so they inherit absolute continuity, i.e. are also given by density function. More precisely given $\varepsilon > 0$, there exist a sequence of points $(x_k) \subset \mathbb{R}^n$ and $R_k \to \infty$ such that for k large enough:

$$\begin{aligned} u_k^{(1)} &:= u_k|_{B_R(x_k)}, & \left| \|u_k^{(1)}\|_{L^2}^2 - \alpha \right| &< \varepsilon, \\ u_k^{(2)} &:= u_k|_{\mathbb{R}^n \setminus B_{R_k}(x_k)}, & \left| \|u_k^{(2)}\|_{L^2}^2 - (1-\alpha) \right| &< \varepsilon. \end{aligned}$$

Restricted functions extended by 0 are not generally in $H^1(\mathbb{R}^n)$ again, therefore we mollify $u_k^{(1)}$ and $u_k^{(2)}$ such that their norms stay the same up to an arbitrary small error and such that the supports lie in a neighbourhood of the original functions.

By scaling with factors close to 1, we obtain

$$\begin{aligned} v_k &:= s_k u_k^{(1)} \in M_\alpha, \quad & s_k^2 &= \tfrac{\alpha}{\|u_k^{(1)}\|_2^2}, \\ w_k &:= t_k u_k^{(2)} \in M_{1-\alpha}, \quad & t_k^2 &= \tfrac{1-\alpha}{\|u_k^{(2)}\|_2^2}. \end{aligned}$$

As $R_k \to \infty$, we get $|E(w_k) - E_\infty(w_k)| \le \varepsilon$ for k large enough. Then we calculate

$$\begin{aligned} I_\alpha + I_{\infty,1-\alpha} &\le E(v_k) + E_\infty(w_k) \\ &\le E(v_k) + E(w_k) + o(1) \\ &= E(v_k + w_k) + o(1) \\ &\le o(1) + s_k^2 \int_{B_R(x_k)} |\nabla u_k|^2\,dx + t_k^2 \int_{\mathbb{R}^n \setminus B_{R_k}(x_k)} |\nabla u_k|^2\,dx \\ &\quad - s_k^p \int_{B_R(x_k)} a(x)|u_k|^p\,dx - t_k^p \int_{\mathbb{R}^n \setminus B_{R_k}(x_k)} a(x)|u_k|^p\,dx \\ &\le o(1) + \Big(1 + o(1)\Big) \int_{\mathbb{R}^n} |\nabla u_k|^2\,dx \\ &\quad - \Big(1 + o(1)\Big) \int_{B_R(x_k)} a(x)|u_k|^p\,dx - \Big(1 + o(1)\Big) \int_{\mathbb{R}^n \setminus B_{R_k}(x_k)} a(x)|u_k|^p\,dx \\ &\le \Big(1 + o(1)\Big) E(u_k) + o(1), \quad \varepsilon \to 0, \end{aligned}$$

where the last inequality used that

$$\hat{C} \ge \int_{B_{R_k}(x_k) \setminus B_R(x_k)} |\nabla u_k|^2\,dx \ge 0$$

as well as $\|u_k - (u_k^{(1)} + u_k^{(2)})\|_2^2 = o(1)$, for $\varepsilon \to 0$ and the inequality $\|u\|_p \le C\|u\|_2^\gamma \|\nabla u\|_2^{1-\gamma}$ from (a), applied to $u_k - (u_k^{(1)} + u_k^{(2)})$. The function $u_k - (u_k^{(1)} + u_k^{(2)})$ equals u_k on $B_{R_K}(x_k) \setminus B_R(x_k)$ and 0 else, so we get

$$\begin{aligned} \Big| \int_{B_{R_k} \setminus B_R(x_k)} a(x)|u_k|^p\,dx \Big| &\le A\|u_k\|^p_{L^p(B_{R_k} \setminus B_R(x_k))} \\ &\le AC\|u_k - (u_k^{(1)} + u_k^{(2)})\|^{p\gamma}_{L^2(\mathbb{R}^n)} \|\nabla u_k\|^{p(1-\gamma)}_{L^2(B_{R_k} \setminus B_R(x_k))} \\ &= o(1), \quad \varepsilon \to 0. \end{aligned}$$

This implies $I_\alpha + I_{1-\alpha,\infty} \le I_1$, a contradiction again.

i) It remains Compactness. By boundedness of (u_k) in $H^1(\mathbb{R}^n)$, we can assume $u_k \xrightarrow{\mathrm{w}} u$ in $H^1(\mathbb{R}^n)$. What we want to show is that the limit function u is an element of M_1.

We know that we can find x_k such that for any $\varepsilon > 0$, there is a radius r with $\int_{B_r(x_k)} u_k^2\,dx \ge 1 - \varepsilon$. First we claim that x_k are bounded. Because assume not, then $|E(u_k) - E_\infty(u_k)| \to \varepsilon$,

$k \to \infty$, due to boundedness of u_k in $H^1(\mathbb{R}^n)$ and as $a(x) \to a_\infty$. Indeed, using $\|u_k\|_p \le C\|u_k\|_2^\gamma \|\nabla u_k\|_2^{1-\gamma}$ again,

$$\begin{aligned}|E(u_k) - E_\infty(u_k)| &= \Big|\int_{\mathbb{R}^n} (a - a_\infty)|u_k|^p\,dx\Big| \\ &\le \sup_{B_r(x_k)} |a - a_\infty| \|u_k\|^p_{L^p(B_r(x_k))} + 2A\|u_k\|^p_{L^p(\mathbb{R}^n \setminus B_r(x_k))} \\ &\le C' \sup_{B_r(x_k)} |a - a_\infty| + O(\varepsilon).\end{aligned}$$

For $\varepsilon \to 0$, this is a contradiction to $I_1 < I_{1,\infty}$.

Therefore, there is $R = R(\varepsilon)$ such that $B_R(0) \supseteq B_r(x_k)$ for all k and so $\int_{B_R(0)} u_k^2\,dx \ge 1 - \varepsilon$. As $u_k \overset{w}{\rightharpoonup} u$ in $H^1(\mathbb{R}^n)$, we get $u_k|_{B_R(0)} \overset{w}{\rightharpoonup} u|_{B_R(0)}$ in $H^1(B_R(0))$. By Rellich's Theorem, $u_k|_{B_R(0)} \to u|_{B_R(0)}$ strongly in $L^2(B_R(0))$.

By the compactness property we get

$$1 - \varepsilon \le \|u_k|_{B_R(0)}\|^2_{L^2(B_R(0))} \le 1$$

and so as well for the limit

$$1 - \varepsilon \le \|u|_{B_R(0)}\|^2_{L^2(B_R(0))} \le 1.$$

If we extend $u|_{B_R(0)}$ by 0 outside $B_R(0)$, then $u|_{B_{R(\varepsilon)}(0)} \to u$ in $L^2(\mathbb{R}^n)$ for $\varepsilon \to 0$ and so $\|u\|_{L^2} = 1$, i.e. $u \in M_1$.

(e) If $a_\infty \le 0$ we first prove that $I_{\lambda,\infty} = 0$ for all λ. $I_{\lambda,\infty} \ge 0$ follows directly from $a_\infty \le 0$. Therefore we just need to construct a sequence u_k s.t. $E_\infty(u_k) \to 0$. Therefore let $u \in M_\lambda$ be any function and define $u_k = k^{-\frac{n}{2}} u(\frac{x}{k})$.The normalization is such that

$$\begin{aligned}\int |u_k|^2\,dx &= \int k^{-n} |u(\tfrac{x}{k})|^2\,dx = \int u(y)\,dy = \lambda, \\ \int |\nabla u_k|^2\,dx &= \int k^{-n-2} |\nabla u(\tfrac{x}{k})|^2\,dx \xrightarrow{k\to\infty} 0.\end{aligned}$$

As in (a) we have that $\|u_k\|_p \le C\|\nabla u_k\|_2^{1-\gamma}$, which tends to 0 for $k \to \infty$ and therefore we get $E_\infty(u_k) \to 0$.

The inequality $I_\lambda < I_{\alpha,\infty} + I_{\lambda-\alpha}$, which is equivalent to the relative compactness of all minimizing sequences, then reads $I_\lambda < I_{\lambda-\alpha}$. Inserting $\alpha = \lambda$ we get $I_\lambda < 0$.

Conversely, assume $I_\lambda < 0$. If $I_{\lambda-\alpha} \ge 0$ we are done, so assume also $I_{\lambda-\alpha} < 0$. Let $u_k \in M_{\lambda-\alpha}$ be a minimising sequence for E on $M_{\lambda-\alpha}$. Let $s^2 = \frac{\lambda}{\lambda-\alpha} > 1$ be the scaling factor such that $su_k \in M_\lambda$. Then

$$\begin{aligned}I_\lambda \le E(su_k) &= \int s^2|\nabla u_k|^2 - as^p|u_k|^p\,dx \\ &= s^p \int s^{2-p}|\nabla u_k|^2 - a|u_k|^p\,dx < s^p E(u_k).\end{aligned}$$

The latter is due to $s^{2-p} < 1$ as $p > 2$. Finally observe that by assumption $E(u_k) < 0$ for large k, as $E(u_k) \to I_{\lambda-\alpha} < 0$. Consequently, $I_\lambda < s^p E(u_k) < E(u_k)$, because $s^p > 1$. The claim $I_\lambda < I_{\lambda-\alpha}$ follows by letting $k \to \infty$.

Remark. If one wants to, one can see that the weak accumulation point $u \in H^1(\mathbb{R}^n)$ obtained in (d) is a minimizer for E on M_1 and the convergence of the (wickedly relabelled) subsequence $(u_k)_k$ to u is actually in the strong $H^1(\mathbb{R}^n)$ topology.

First the proof of the minimality of u which amounts to showing $E(u) \leq \liminf_{k\to\infty} E(u_k)$: Since ∇u_k converges weakly to ∇u in $L^2(\mathbb{R}^n)$, the w.s.l.s.c. of the norm implies

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx.$$

For the second part, we show convergence: for arbitrary $\varepsilon > 0$, take $R > 0$ such that $\int_{B_R(0)} |u_k|^2 \, dx \geq 1 - \varepsilon$ (as above). Due to Rellich's compactness theorem, $u_k \to u$ in $L^p(B_R(0))$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \int_{B_R(0)} a(x)|u_k|^p \, dx = \int_{B_R(0)} a(x)|u|^p \, dx.$$

Further,

$$\begin{aligned} \Big| \int_{B_R(0)^c} a(x)|u_k|^p \, dx \Big| &\leq \|a\|_\infty C^p \|u_k\|^{p\gamma}_{L^2(B_R(0)^c)} \|\nabla u_k\|_2^{p(1-\gamma)} \\ &\leq \|a\|_\infty C^p \varepsilon^{\frac{p\gamma}{2}} \underbrace{\sup_{l\in\mathbb{N}} \|\nabla u_l\|_2^{p(1-\gamma)}}_{<\infty}, \end{aligned}$$

where the same interpolation has been used as before. The same estimate holds true for u, which leads to

$$\limsup_{k\to\infty} \Big| \int_{\mathbb{R}^n} a(x)|u_k|^p \, dx - \int_{\mathbb{R}^n} a(x)|u|^p \, dx \Big| \leq C' \varepsilon^{\frac{p\gamma}{2}}.$$

Since $\varepsilon > 0$ was arbitrary, this leaves us with

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} a(x)|u_k|^p \, dx = \int_{\mathbb{R}^n} a(x)|u|^p \, dx.$$

Finally, we arrive at $E(u) \leq \liminf_{k\to\infty} E(u_k) = I_1$, proving that u is indeed a minimizer of E on the set M_1.

Now, making use of $E(u_k) \to E(u)$ as $k \to \infty$ (as seen above) and the just shown convergence $\int_{\mathbb{R}^n} a(x)|u_k|^p \, dx \to \int_{\mathbb{R}^n} a(x)|u|^p \, dx$, we get that $\|\nabla u_k\|^2_{L^2(\mathbb{R}^n)} \to \|\nabla u\|^2_{L^2(\mathbb{R}^n)}$ as $k \to \infty$. Together with the weak L^2-convergence of ∇u_k towards ∇u, this proves strong L^2-convergence of ∇u_k towards ∇u. Strong convergence of u_k towards u in $L^2(\mathbb{R}^n)$ can be obtained by a diagonal argument using $u_k|_{B_R(0)} \to u|_{B_R(0)}$ strongly in $L^2(B_R(0))$ and $u|_{B_R(0)} \to u$ strongly in $L^2(\mathbb{R}^n)$.

This finishes the proof of the statement that u is even an accumulation point of the minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in the strong H^1 topology.