Solution 5

1. Concentration – Compactness.

(a) As $E(\alpha u) = \alpha^2 E(u)$ for any function u we have $I_\lambda = \lambda^2 I_1$ and $I_{1-\lambda} = (1-\lambda)^2 I_1$. So

$$\begin{aligned} I_\lambda + I_{1-\lambda} &= (\lambda^2 + (1-\lambda)^2) I_1 \\ &= (1 - 2\lambda + 2\lambda^2) I_1. \end{aligned}$$

If $\lambda \in (0,1)$, we have $(1 - 2\lambda + 2\lambda^2) < 1$ and therefore we get

$$I_\lambda + I_{1-\lambda} \begin{cases} < I_1, & \text{if } I_1 > 0 \\ > I_1, & \text{if } I_1 < 0 \\ = I_1, & \text{if } I_1 = 0. \end{cases}$$

Finally we note that $I_1 < 0$ if and only if $I_\lambda < 0$ for all $\lambda > 0$.

(b) Note first that $E(u)$ is bounded from below for $u \in M_1$, because the term

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) u(y)\, \chi_{B_R(0)}(x-y)\, dx\, dy$$

is bounded by 1. Let $u_k \in M_1$ be a minimising sequence for E. Then $d\mu_k = u_k\, dx$ are probability measures, so Theorem 1.3.2. from the lecture (the Concentration – Compactness - Lemma) gives a subsequence (still denoted by u_k) satisfying either i) Compactness, ii) Vanishing or iii) Dichotomy.

ii) Assume Vanishing, i.e. u_k are nowhere concentrated. Then, as R from the definition of the functional E is a fixed number, we get that $E(u_k) = \int_{\mathbb{R}^n} u_k^2\, dx + o(1)$ due to

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u_k(x) u_k(y)\, \chi_{B_R(0)}(x-y)\, dx\, dy = \int_{\mathbb{R}^n} u(y) \int_{B_R(y)} u(x)\, dx\, dy \\ &\le \|u_k\|_{L^1(\mathbb{R}^n)} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} u_k(x)\, dx \xrightarrow{k \to \infty} 0. \end{aligned}$$

So for $k \to \infty$ we see that we have $I_1 \ge 0$, which is a contradiction (as seen in (a)); so this case cannot occur.

iii) Assume Dichotomy with parameter $\lambda \in (0,1)$. Then there are r, $x_k \in \mathbb{R}^n$ and $r_k \to \infty$ such that for $u_k^{(1)} = u_k|_{B_r(x_k)}$ and $u_k^{(2)} = u_k|_{\mathbb{R}^n \setminus B_{r_k}(x_k)}$ we have

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} u_k^{(1)}\, dx - \lambda \Big| &< \varepsilon, \\ \Big| \int_{\mathbb{R}^n} u_k^{(2)}\, dx - (1-\lambda) \Big| &< \varepsilon. \end{aligned}$$

For large enough k the supports of $u_k^{(1)}$ and $u_k^{(2)}$ are at least distance R apart, so that $E(u_k^{(1)} + u_k^{(2)}) = E(u_k^{(1)}) + E(u_k^{(2)})$.

As the norms of $u_k^{(i)}$ are not exactly what we want, we define

$$
\begin{aligned}
v_k &:= u_k^{(1)} \tfrac{\lambda}{\|u_k^{(1)}\|_1} \in M_\lambda, \\
w_k &:= u_k^{(2)} \tfrac{1-\lambda}{\|u_k^{(2)}\|_1} \in M_{1-\lambda}.
\end{aligned}
$$

Note that $\frac{\lambda}{\|u_k^{(1)}\|_1} = 1 + o(1)$ for $\varepsilon \to 0$.

Using homogeneity of the functional E we get that $E(u_k^{(1)}) - E(v_k) = \Big(1 - \big(\frac{\lambda}{\|u_k^{(1)}\|}\big)^2\Big) E(u_k)$, and similarly for $u_k^{(2)}$ and w_k.

We can calculate

$$
\begin{aligned}
&E(u_k^{(1)}) + E(u_k^{(2)}) - E(u_k) \\
&\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big(u_k(x)u_k(y) - u_k^{(1)}(x)u_k^{(1)}(y) - u_k^{(2)}(x)u_k^{(2)}(y)\Big)\, \chi(x-y)\, dx\, dy \\
&= \int_{\mathbb{R}^n} u_k(y) \int_{\mathbb{R}^n} u_k(x)\, \chi(x-y)\, dx \\
&\quad - u_k^{(1)}(y) \int_{B_r} u_k(x)\, \chi(x-y)\, dx - u_k^{(2)}(y) \int_{\mathbb{R}^n \setminus B_{r_k}} u_k(x)\, \chi(x-y)\, dx\, dy \\
&= \int_{\mathbb{R}^n} u_k(y) \int_{B_{r_k} \setminus B_r} u_k(x)\, \chi(x-y)\, dx \\
&\quad + \Big(u_k - u_k^{(1)}\Big)(y) \int_{B_r} u_k(x)\, \chi(x-y)\, dx + \Big(u_k - u_k^{(2)}\Big)(y) \int_{\mathbb{R}^n \setminus B_{r_k}} u_k(x)\, \chi(x-y)\, dx\, dy \\
&\le 3\|u_k\|_{L^1(B_{r_k} \setminus B_r)}.
\end{aligned}
$$

The last estimate is due to

$$
\begin{aligned}
&\operatorname{supp}\Big(u_k - u_k^{(1)}\Big) \subset \mathbb{R}^n \setminus B_r, \qquad \operatorname{supp}\Big(\int_{B_r} u_k(x)\, \chi(x-y)\, dx\Big) \subset B_{r+R} \subset B_{r_k}, \\
&\operatorname{supp}\Big(u_k - u_k^{(2)}\Big) \subset B_{r_k}, \qquad \operatorname{supp}\Big(\int_{\mathbb{R}^n \setminus B_{r_k}} u_k(x)\, \chi(x-y)\, dx\Big) \subset \mathbb{R}^n \setminus B_{r_k - R} \subset \mathbb{R}^n \setminus B_r.
\end{aligned}
$$

From this it follows

$$
\begin{aligned}
I_\lambda + I_{1-\lambda} &\le \limsup_{k\to\infty} \Big(E(v_k) + E(w_k)\Big) \\
&= \limsup_{k\to\infty} (1 + o(1)) \Big(E(u_k^{(1)}) + E(u_k^{(2)})\Big) \le (1 + o(1)) I_1 + 6\varepsilon.
\end{aligned}
$$

This holds for all $\varepsilon > 0$, so we get again a contradiction to $I_\lambda + I_{1-\lambda} > I_1$.

i) Therefore it holds Compactness, i.e. $\int_{B_r(x_k)} u_k\, dx > 1 - \varepsilon$ for some $x_k \in \mathbb{R}^n$ and r large enough. We cannot exclude that $|x_k| \to \infty$, but if we consider $v_k = u_{k,x_k}$, the translated function, then we see that $E(v_k) = E(u_k)$, as E is translation invariant, i.e. translating the u_k anywhere, we still have a minimising sequence. So we can assume that the functions u_k are concentrated around 0. (We only want to show the existence of a minimiser, not the relative compactness of all minimising sequences.) We then take a weakly converging subsequence on

$B_r(0)$ and letting $r \to \infty$ and taking further subsequences we get a weak limit u with norm 1. We further need that $u \geq 0$ to be an element of M_1. This follows from testing for example with χ_K, where $K = \{x \mid u(x) < 0\}$.

As the u_k are a minimising sequence (and as $E \not\equiv \infty$), we have that $\int_{\mathbb{R}^n} u_k^2 \, dx < C < \infty$ and so there is a further subsequence of the u_k converging weakly in L^2 as well. The L^2-norm is w.s.l.s.c. Indeed, for any dual f

$$\|u_k\| \|f\| \geq (u_k, f) \to (u, f) \qquad \Rightarrow \|u\| \leq \liminf_{k\to\infty} \|u_k\|.$$

It remains to deal with the term $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y) u(x) \, \chi_{B_R(0)}(x-y) \, dx \, dy$. Using the Camel trick[1] with $u_k(y) \int_{B_R(y)} u(x) \, dx$ we obtain,

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_k(y) \int_{\mathbb{R}^n} u_k(x) \, \chi_{B_R(0)}(x-y) \, dx \, dy - \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} u(x) \, \chi_{B_R(0)}(x-y) \, dx \, dy \\
&= \int_{\mathbb{R}^n} \Big(u_k(y) \underbrace{\int_{B_R(y)} (u_k - u)(x) \, dx}_{g_k(y)} \Big) dy - \int_{\mathbb{R}^n} (u - u_k)(y) \, (u * \chi_{B_R})(y) \, dy.
\end{aligned}$$

The convolution $(u * \chi_{B_R})$ of $u \in L^2(\mathbb{R}^n)$ with $\chi_{B_R} \in L^1(\mathbb{R}^n)$ is again in $L^2(\mathbb{R}^n)$ by Young's inequality $\|u * \chi_{B_R}\|_{L^2} \leq \|u\|_{L^2} \|\chi_{B_R}\|_{L^1}$. It therefore serves as test function for the weak L^2-convergence $u_k \overset{\text{w}}{\rightharpoonup} u$ and leads to

$$\int_{\mathbb{R}^n} (u - u_k)(y) \, (u * \chi_{B_R})(y) \, dy \xrightarrow{k\to\infty} 0.$$

Testing $u_k \overset{\text{w}}{\rightharpoonup} u$ with $\chi_{B_R(y)} \in L^2(\mathbb{R}^n)$ also shows that $g_k(\cdot)$ defined above converges pointwise to 0 but not necessarily in L^2. However, $2 \geq |g_k(y)|$ serves as integrable majorant when restricting y to a bounded domain. Thus, let $\varepsilon > 0$ be fixed and $r > 0$ the corresponding compactness radius. By dominated convergence, $\|g_k\|_{L^2(B_r(0))} \to 0$ as $k \to \infty$. The scalar products of weakly convergent functions with strongly convergent ones converge to the scalar product of the respective limits. Therefore,

$$\int_{B_r(0)} u_k(y) \, g_k(y) \, dy \xrightarrow{k\to\infty} 0.$$

On the complement, we appeal to compactness and estimate

$$\int_{\mathbb{R}^n \setminus B_r(0)} u_k(y) \, g_k(y) \, dy \leq \Big(\sup_{\mathbb{R}^n \setminus B_r(0)} |g_k| \Big) \int_{\mathbb{R}^n \setminus B_r(0)} u_k \, dy \leq 2\varepsilon.$$

As a result,

$$E(u) \leq \liminf_{k\to\infty} E(u_k) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $(u_k)_k$ a minimising sequence for E on M_1 we conclude, that the minimum is in fact attained at $u \in M_1$.

[1] Malba Tahan. "The man who counted – A Collection of Mathematical Adventures", §3 Beasts of Burden.

(c) Consider the function

$$u(x) = \frac{1}{\mathcal{L}^n(B_2(0))}\chi_{B_2(0)}(x).$$

Then

$$\int_{\mathbb{R}^n} u^2\,dx = \frac{1}{\mathcal{L}^n(B_2(0))},$$
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)u(y)\chi_{B_R(0)}(x-y)\,dx\,dy = \left(\int_{\mathbb{R}^n} u(x)\,dx\right)\left(\int_{\mathbb{R}^n} u(y)\,dy\right) = 1,\ \text{if } R > 4.$$

As $\frac{1}{\mathcal{L}^n(B_2(0))} < 1$, we see that $E(u) < 0$, if $R > 4$.