Solution 6

1. Baby-Yamabe-Problem.

(a) If $\lambda \leq 0$, coercivity is obvious. Assume therefore $\lambda > 0$.

For all $u \in C_c^\infty(\Omega)$ it holds $\|u\|_2^2 \leq \frac{\|\nabla u\|_2^2}{\lambda_1}$. By density of $C_c^\infty(\Omega) \subseteq H_0^1(\Omega)$ the same holds for $u \in H_0^1(\Omega)$.

Given any $u \in H_0^1(\Omega)$ we therefore get

$$\begin{aligned} E_\lambda(u) &= \|\nabla u\|_2^2 - \lambda \|u\|_2^2 \\ &\geq \left(1 - \tfrac{\lambda}{\lambda_1}\right) \|\nabla u\|_2^2, \end{aligned}$$

where the inequality holds, because $\lambda > 0$. If now $\lambda < \lambda_1$, this tends to ∞, if $\|u\|_{H_0^1} \to \infty$.

(b) In the following, we assume for simplicity $0 \in \Omega$. (Which we can by translating the coordinate system.) Let $u \in C_c^\infty(\Omega)$ be any function and for $k \in \mathbb{N}$ large enough define $v_k(x) := u(kx) \in C_c^\infty(\Omega)$. ($k$ needs to be large enough to ensure that the support of v_k is indeed a subset of Ω.) Then we get

$$\begin{aligned} \|v_k\|_2^2 &= \int_\Omega |u(kx)|^2 \, dx = \int_\Omega |u(y)|^2 k^{-n} \, dy \\ &= k^{-n} \|u\|_2^2 \\ \|\nabla v_k\|_2^2 &= \int_\Omega |\nabla u(kx)|^2 \, dx = \int_\Omega k^2 |(\nabla u)(kx)|^2 \, dx = \int_\Omega |\nabla u(y)|^2 k^{2-n} \, dy \\ &= k^{2-n} \|\nabla u\|_2^2 \\ \|v_k\|_{2^*}^2 &= \Big(\int_\Omega |u(kx)|^{2^*} \, dx\Big)^{\frac{2}{2^*}} = \Big(\int_\Omega |u(y)|^{2^*} k^{-n} \, dy\Big)^{\frac{2}{2^*}} \\ &= k^{-\frac{2n}{2^*}} \|u\|_{2^*}^2. \end{aligned}$$

Therefore, using $2 - n = -\frac{2n}{2^*}$:

$$\begin{aligned} \frac{E_\lambda(v_k)}{\|v_k\|_{2^*}^2} &= \frac{k^{2-n}\|\nabla u\|_2^2 - \lambda k^{-n} \|u\|_2^2}{k^{-\frac{2n}{2^*}} \|u\|_{2^*}^2} \\ &= \frac{E_0(u)}{\|u\|_{2^*}^2} - \frac{\lambda k^{-n} \|u\|_2^2}{k^{-\frac{2n}{2^*}} \|u\|_{2^*}^2}. \end{aligned}$$

As $n > \frac{2n}{2^*}$, the last term tends to 0 for $k \to \infty$.

By Hölder, the last term is bounded, independently of u. If we now let u_m be a sequence minimising S_0, we can find $k(m)$, such that

$$\frac{E_\lambda(v_{m,k(m)})}{\|v_{m,k(m)}\|_{2^*}^2} \leq \frac{E_0(u_m)}{\|u_m\|_{2^*}^2} + \frac{1}{m}.$$

From this, the first claim follows.

If now $\lambda \leq 0$, we see that $E_\lambda(u) \geq E_0(u)$ for all $u \in H_0^1(\Omega)$. Therefore we also get the inequality $S_\lambda \geq S_0$.

(c) Let v be the first Laplace eigenfunction, i.e. $\lambda_1 = \frac{\|\nabla v\|_2^2}{\|v\|_2^2}$. Then

$$\begin{aligned}
&E_\lambda(v) = \|\nabla v\|_2^2 - \lambda \|v\|_2^2 = \left(\lambda_1 - \lambda\right) \|v\|_2^2 \\
&\Rightarrow S_\lambda \leq \frac{E_\lambda(v)}{\|v\|_{2^*}^2} = \left(\lambda_1 - \lambda\right) \frac{\|v\|_2^2}{\|v\|_{2^*}^2}.
\end{aligned}$$

As $\frac{\|v\|_2^2}{\|v\|_{2^*}^2}$ is a constant not depending on λ, we see that this term converges to 0 if $\lambda \to \lambda_1$. As $S_0 > 0$, we can make it smaller than S_0, by requiring that $\lambda_1 - \lambda < S_0 \frac{\|v\|_{2^*}^2}{\|v\|_2^2}$.

(d) Assume $S_\lambda = S_0$.

Let first $\lambda < 0$, in which case $S_\lambda = S_0$ is always satisfied. Assume $u_k \in M$ is a minimizing sequence for E_λ. Then

$$\begin{aligned}
S_\lambda + o(1) = E_\lambda(u_k) &= E_0(u_k) + \lambda \|u_k\|_2^2 \\
&\geq S_0 + \lambda \|u_k\|_2^2 = S_\lambda + \lambda \|u_k\|_2^2, \quad k \to \infty.
\end{aligned}$$

So we need to have that for every convergent sequence $\|u_k\|_2^2 \to 0$, i.e. $u_k \to 0$ in L^2. So in particular there is no limit in M.

Assume $\lambda = 0$. Then there cannot be any convergent minimising sequence in M. Because assume there was $u \in M$ with $E_0(u) = S$. Then we scale u as $v_k(x) = k^{\frac{n-2}{2}} u(kx)$. With the calculations from (b) we get

$$\begin{aligned}
&\|v_k\|_{2*}^2 = \|u\|_{2^*}^2 \\
&E_0(v_k) = E_0(u) = S.
\end{aligned}$$

Each of these v_k is therefore an element of M and as they are all minimisers E_0, there are numbers $\alpha_k > 0$ such that $w_k = \alpha_k v_k$ are solutions of

$$\begin{cases} -\Delta w_k = w_k |w_k|^{2^*-2} & \text{in } \Omega, \\ \quad w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

See Lemma 1.3.2. from the lecture for a proof of this claim.

Using regularity theory for the Laplace operator, we get that $w_k \in C^1(\Omega)$. But for $k > 1$, the support of w_k is a proper subset of Ω and by the strong maximum principle proven in Problem Set 3, $w_k \equiv 0$, i.e. $u \equiv 0$, which is a contradiction.

Finally let $\lambda > 0$. Let u_k be any minimising sequence for E_0 in M. Then

$$\begin{aligned}
S_0 + o(1) = E_0(u_k) &= E_\lambda(u_k) + \lambda \|u_k\|_2^2 \\
&\geq E_\lambda(u_k) \geq S_\lambda = S_0, \quad k \to \infty,
\end{aligned}$$

which implies that u_k is a minimising sequence for E_λ as well. But we have shown above that this sequence cannot have a convergent subsequence in M, so we have found a minimising sequence for E_λ, which is not relatively compact in M.

Assume now $S_\lambda < S_0$.

Let $u_k \in M$ be a sequence minimising E_λ on M. By (a) we know that $(u_k)_k$ is bounded in H_0^1, so there is subsequence which converges weakly in H_0^1. As $\|u_k\|_{2^*} = 1$, there is a further subsequence converging weakly in L^{2^*}, too. The unit ball in L^{2^*} is weakly closed (because it is norm closed and convex), so we get that the limit $v \in H_0^1(\Omega)$ satisfies $0 \leq \|v\|_{2^*} =: \alpha \leq 1$. Note that the limits of u_k in L^{2^*} agrees with the limit in H_0^1, because $(H_0^1)^* \supseteq (L^{2^*})^*$.

If $\alpha = 1$, we are done, because then $v \in M$ i.e. we have found a subsequence converging in M. So we just need to rule out $\alpha < 1$.

If $\alpha = 0$, this implies $u_k \xrightarrow{w} 0$. By Rellich's Theorem then $u_k \to 0$ strongly in $L^2(\Omega)$, which implies

$$\begin{aligned} E_\lambda(u_k) &= \|\nabla u_k\|_2^2 - \lambda \|u_k\|_2^2 = \|\nabla u_k\|_2^2 + o(1) \\ &= E_0(u_k) + o(1), \quad k \to \infty. \end{aligned}$$

But from this we conclude $S_0 \leq S_\lambda$, a contradiction.

If $\alpha \in (0,1)$, we use Lemma 1.3.1. and Rellich's Theorem ($u_k \to v$ strongly in $L^2(\Omega)$) to calculate

$$\begin{aligned} E_\lambda(u_k) &= \|\nabla u_k\|_2^2 - \lambda \|u_k\|_2^2 \\ &= \|\nabla v\|_2^2 + \|\nabla(v - u_k)\|_2^2 + o(1) - \lambda\Big(\|v\|_2^2 + \underbrace{\|v - u_k\|_2^2}_{\to 0} + o(1)\Big) \\ &= E_\lambda(v) + E_0(v - u_k) + o(1), \quad k \to \infty. \end{aligned}$$

Lemma 1.3.1. tells us also

$$\begin{aligned} &\|u_k\|_{2^*}^{2^*} = \|v\|_{2^*}^{2^*} + \|v - u_k\|_{2^*}^{2^*} + o(1), \quad k \to \infty, \\ \Rightarrow\ &\|v - u_k\|_{2^*}^{2^*} = 1 - \alpha^{2^*} + o(1). \end{aligned}$$

Combining these and using $\frac{E_\lambda(w)}{\|w\|_{2^*}^2} \geq S_\lambda$ for all $w \neq 0$, we get

$$\begin{aligned} S_\lambda &= E_\lambda(u_k) + o(1) \\ &= E_\lambda(v) + E_0(v - u_k) + o(1) \\ &\geq \alpha^2 S_\lambda + \|v - u_k\|_{2^*}^2 S_0 + o(1) \\ &\geq \alpha^2 S_\lambda + (1 - \alpha^{2^*} + o(1))^{\frac{2}{2^*}} S_0 + o(1) \\ &\geq \alpha^2 S_\lambda + (1 - \alpha^{2^*})^{\frac{2}{2^*}} S_0 + o(1), \quad k \to \infty. \end{aligned}$$

$$\Rightarrow (1 - \alpha^2) S_\lambda \geq (1 - \alpha^{2^*})^{\frac{2}{2^*}} S_0 > (1 - \alpha^2) S_0,$$

where the last inequality holds because $\alpha^{2^*} < \alpha^2$. So we arrived at a contradiction to the condition $S_\lambda < S_0$.