Solution 7

1. Hodge $*$-Operator.

(a) Note first that $*\det(u) = *(\det(u)\cdot 1) = \det(u)\wedge(*\,1) = \det(u)\,dx^1\wedge\ldots\wedge dx^n$.

We will give two versions how to prove this exercise.

Version 1: Let $e_1,\ldots,e_n$ be the dual basis of $dx^1,\ldots,dx^n$. Then

$$du^1\wedge\ldots\wedge du^n = (du^1\wedge\ldots\wedge du^n)(e_1,\ldots,e_n)\,dx^1\wedge\ldots\wedge dx^n$$

and this coefficient is exactly the determinant:

$$\begin{aligned}(du^1\wedge\ldots\wedge du^n)(e_1,\ldots,e_n) &= \sum_{\sigma\in\mathcal{S}^n}\Big(\operatorname{sign}(\sigma)(du^{\sigma(1)}\otimes\ldots\otimes du^{\sigma(n)})(e^1,\ldots,e^n)\Big)\\ &= \sum_{\sigma\in\mathcal{S}^n}\Big(\operatorname{sign}(\sigma)\prod_{i=1}^n du^{\sigma(i)}(e_i)\Big)\\ &= \sum_{\sigma\in\mathcal{S}^n}\Big(\operatorname{sign}(\sigma)\prod_{i=1}^n \frac{\partial u^{\sigma(i)}}{\partial x^i}\Big)\\ &= \det(du).\end{aligned}$$

Version 2: In the lecture we have seen that

$$du^k\wedge du^l = \sum_{i<j}\det(A^{kl}_{ij})\,dx^i\wedge dx^j,$$

where $A^{kl}_{ij} = \begin{pmatrix}\partial_i u^k & \partial_j u^k\\ \partial_i u^l & \partial_j u^l\end{pmatrix}$. From this, the case $n=2$ followed directly. To conclude the cases $n\geq 2$, we proceed by induction. For simplicity of notation, we will in the following show the step $2\mapsto 3$. The general case $n\mapsto n+1$ works similarly.

We take the product with du^m:

$$du^k\wedge du^l\wedge du^m = \sum_{i<j}\sum_s \det(A^{kl}_{ij})\partial_s u^m\,dx^i\wedge dx^j\wedge dx^s.$$

To order this in a similar way as for $n=2$, we consider some fixed $a<b<c$. We have three terms corresponding to this, namely if (a,b,c) is one of (i,j,s), (i,s,j) or (s,i,j). The corresponding terms are

$$\begin{aligned}&\det(A^{kl}_{ab})\partial_c u^m\,dx^a\wedge dx^b\wedge dx^c + \det(A^{kl}_{ac})\partial_b u^m\,dx^a\wedge dx^c\wedge dx^b +\\ &\qquad + \det(A^{kl}_{bc})\partial_a u^m\,dx^b\wedge dx^c\wedge dx^a\\ &= \Big(\det(A^{kl}_{ab})\partial_c u^m - \det(A^{kl}_{ac})\partial_b u^m + \det(A^{kl}_{bc})\partial_a u^m\Big)\,dx^a\wedge dx^b\wedge dx^c\\ &= \det(A^{klm}_{abc})\,dx^a\wedge dx^b\wedge dx^c,\end{aligned}$$

where $A^{klm}_{abc} = \begin{pmatrix} \partial_a u^k & \partial_b u^k & \partial_c u^k \\ \partial_a u^l & \partial_b u^l & \partial_c u^l \\ \partial_a u^m & \partial_b u^m & \partial_c u^m \end{pmatrix}$ and where the last step used Laplace's formula for determinants, because A^{kl}_{ab}, A^{kl}_{ac} and A^{kl}_{bc} are all minors of A^{klm}_{abc}. Therefore

$$du^k \wedge du^l \wedge du^m = \sum_{a<b<c} \det(A^{klm}_{abc})\, dx^a \wedge dx^b \wedge dx^c.$$

From this, the case $n = 3$ follows directly and we can continue the induction to prove the claim for all n.

(b) By definition of the Hodge $*$-operator, $*\, dx^i = (-1)^{i+1}\, dx^1 \wedge \ldots \wedge \hat{dx^i} \wedge \ldots \wedge dx^n$, where $\hat{dx^i}$ means omitting dx^i. This is because if we let in the definition $b = dx^i$ and test with $a = dx^j$ we get

$$dx^j \wedge *\, dx^i = *(dx^j \cdot dx^i) = *(\delta_{ij}) = \delta_{ij} dx^1 \wedge \ldots \wedge dx^n.$$

Then we can calculate

$$\begin{aligned} d^*\Big(\sum_i a_i dx^i\Big) &= *\, d * \Big(\sum_i a_i dx^i\Big) \\ &= *\, d\Big(\sum (-1)^{i+1} a_i dx^1 \wedge \ldots \wedge \hat{dx^i} \wedge \ldots \wedge dx^n\Big) \\ &= *\Big(\sum_i (-1)^{i+1} \frac{\partial a_i}{\partial x^i} dx^i \wedge dx^1 \wedge \ldots \wedge \hat{dx^i} \wedge \ldots \wedge dx^n\Big) \\ &= *\Big(\sum_i \frac{\partial a_i}{\partial x^i} dx^1 \wedge \ldots \wedge dx^n\Big) \\ &= \sum_i \frac{\partial a_i}{\partial x^i} \\ &= \operatorname{div}(a_1, \ldots, a_n) \end{aligned}$$

(c) First we calculate

$$\begin{aligned} d^* f &= *\, d * f \\ &= *\, d(f\, dx^1 \wedge \ldots \wedge dx^n) \\ &= *(df \wedge dx^1 \wedge \ldots \wedge dx^n) = 0, \end{aligned}$$

therefore $\Delta f = d^* df$. As $df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n$ is a 1-form we can apply (b) which tells us

$$\Delta f = d^* df = \operatorname{div}\Big(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\Big),$$

which is the usual Laplacian of f.

2. Area Functional.

(a) Let $\Omega \subset\subset \mathbb{R}^n$ with $n > 1$. The argument given in the lecture shows that the area functional on $W^{1,1}(\Omega)$ is not differentiable at constant u. Let $u \in W^{1,1}(\Omega)$ enjoy $\|\nabla u\|_{L^1(\Omega)} > 0$ and let $g\colon \Omega \to \mathbb{R}$ represent $\nabla u \in L^1(\Omega)$. The set

$$S_b = \{x \in \Omega \mid |g(x)| > C_b := \tfrac{1}{b|\Omega|}\|\nabla u\|_{L^1(\Omega)}\}$$

is defined for $b \in (0,1)$ and has measure $|S_b| \le b|\Omega|$ due to

$$\|\nabla u\|_{L^1(\Omega)} = \int_\Omega |g|\,dx \ge \int_{S_b} |g|\,dx \ge \frac{|S_b|}{b|\Omega|}\|\nabla u\|_{L^1(\Omega)}.$$

Thus, the complement $S_b^\complement = \Omega \setminus S_b$ has measure $|S_b^\complement| \ge (1-b)|\Omega|$. For any sufficiently small radius $r > 0$, there exists $x_0 \in S_b^\complement$ such that $|B_r(x_0) \cap S_b^\complement| \ge (1-b)|B_r|$. We consider the following function v_k, its gradient and the given integrals while denoting the n-volume of the unit ball $B_1 \subset \mathbb{R}^n$ by ω_n.

$$0 \le v_k(x) = \big(1 - k|x|\big)\chi_{B_{\frac{1}{k}}} \le 1, \qquad |\nabla v_k(x)| = k\,\chi_{B_{\frac{1}{k}}},$$

$$\int_{B_{\frac{1}{k}}} |v_k|\,dx = n\omega_n \int_0^{\frac{1}{k}} \big(1 - kr\big) r^{n-1}\,dr \le \omega_n k^{-n}, \qquad \int_{B_{\frac{1}{k}}} |\nabla v_k|\,dx = \omega_n k^{-n+1}.$$

The function $f(x) = \sqrt{1 + |x|^2}$ being convex on $\mathbb{R}^n$ satisfies $f(x+h) - f(x) \ge \nabla f(x)\cdot h$.

$$\Rightarrow\ D(u,v) := \sqrt{1 + |\nabla u + \nabla v|^2} - \sqrt{1 + |\nabla u|^2} - \frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}} \ge 0$$

for any $u, v \in W^{1,1}(\Omega)$. Therefore,

$$\int_\Omega D(u,v)\,dx \ge \int_{B_{\frac{1}{k}} \cap S_b^\complement} D(u,v)\,dx.$$

We may adapt the coordinate system (each time k) such that x_0 is the origin. Since $|\nabla u| \le C_b$ holds at almost every $x \in B_{\frac{1}{k}} \cap S_b^\complement$ we may estimate

$$\frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}} \le \frac{C_b\,k}{\sqrt{1 + C_b^2}} =: \delta_b k,$$

$$|\nabla u + \nabla v|^2 \ge |\nabla u|^2 - 2|\nabla u||\nabla v| + |\nabla v|^2 \ge -2C_b k + k^2,$$

$$\sqrt{1 + |\nabla u + \nabla v|^2} - \sqrt{1 + |\nabla u|^2} \ge \sqrt{1 - 2C_b k + k^2} - \sqrt{1 + C_b^2} \ge k - \sqrt{2C_b k} - \sqrt{1 + C_b^2}$$

using concavity crudely via $\sqrt{1 - x + y} \ge \sqrt{y - x} \ge \sqrt{y} - \sqrt{x}$. Note that $\delta_b < 1$ for b fixed.

$$\Rightarrow\ \int_{B_{\frac{1}{k}} \cap S_b^\complement} D(u, v_k)\,dx \ge (1-b)|B_1|k^{-n}\Big((1-\delta_b)k - \sqrt{2C_b k} - \sqrt{1 + C_b^2}\Big)$$

which asymptotically decreases not faster than k^{-n+1}. as $k \to \infty$. However, $\|v_k\|_{W^{1,1}} = O(k^{-n+1})$ as $k \to \infty$ implying that

$$\frac{E(u + v_k) - E(u) - dE(u)v_k}{\|v_k\|_{W^{1,1}(\Omega)}}$$

does not converge to zero as $k \to \infty$.

(b) If $u \mapsto dE(u)$ was continuous in a neighbourhood of u_0, then E would be Fréchet-differentiable at u_0. But this was ruled out above, so there are points of discontinuouity.