# Solution 8

**1. Präsenzaufgabe.**

**(a)** As $\langle dE(v), v\rangle = \|\nabla v\|_2^2 - \|v\|_p^p$ for all $v \in H_0^1(\Omega)$, we need to find a zero of the continuous map $f\colon (0,1) \to \mathbb{R}$, $s \mapsto \|\nabla\gamma(s)\|_2^2 - \|\gamma(s)\|_p^p$. As $E(y(1)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p < 0$, we have that for $s$ close to 1: $f(s) < 0$.

By Sobolev embedding we have $\|\gamma(s)\|_p^p \leq c\|\nabla\gamma(s)\|_2^p$ and therefore

$$\begin{aligned} f(s) &\geq \|\nabla\gamma(s)\|_2^2 - c\|\nabla\gamma(s)\|_2^p \\ &= \|\nabla\gamma(s)\|_2^2\Big(1 - c\|\nabla\gamma(s)\|_2^{p-2}\Big). \end{aligned}$$

As $\gamma(0) = 0$ we see that $f(s) > 0$ for $s$ small enough. Using the mean value Theorem we then find some $s_\gamma \in (0,1)$ with $f(s_\gamma) = 0$, i.e. $\|\nabla\gamma(s_\gamma)\|_2^2 = \|\gamma(s_\gamma)\|_p^p$.

**(b)** We can find the supremum by differentiating the map $\lambda \mapsto E(\lambda v)$.

$$\begin{aligned} E(\lambda v) &= \frac{1}{2}\lambda^2\|\nabla v\|_2^2 - \frac{1}{p}\lambda^p\|v\|_p^p \\ &= \frac{1}{2}\lambda^2\|\nabla v\|_2^2 - \frac{1}{p}\lambda^p \\ \Rightarrow \frac{d}{d\lambda}E(\lambda v) &= \lambda\|\nabla v\|_2^2 - \lambda^{p-1}. \end{aligned}$$

$\lambda = 0$ gives the local minimum 0, so we can assume $\lambda \neq 0$ and then we get $\lambda = \|\nabla v\|_2^{\frac{2}{p-2}}$. This needs to be the maximum, because $E(\lambda v) > 0$ for small $\lambda$ whereas $E(\lambda v) \to -\infty$, $\lambda \to \infty$. Therefore

$$\begin{aligned} \sup_{0<\lambda<\infty} E(\lambda v) &= \frac{1}{2}\|\nabla v\|_2^{\frac{4}{p-2}}\|\nabla v\|_2^2 - \frac{1}{p}\|\nabla v\|_2^{\frac{2p}{p-2}} \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\nabla v\|_2^{\frac{2p}{p-2}}. \end{aligned}$$

**(c)** To prove this identity we will show that for all $\gamma \in \Gamma$ it holds:

$$\sup_{0<\lambda} E(\lambda\bar{u}) \leq \sup_{0\leq s\leq 1} E(\gamma(s)). \tag{1}$$

From this we get the inequality "$\leq$" . The other inequality follows when noting that for $N$ large enough, $s \mapsto sN\bar{u}$ is a path in $\Gamma$.

To prove (1) we calculate on one hand

$$\sup_{0<\lambda} E(\lambda\bar{u}) = \frac{p-2}{2p}\|\nabla\bar{u}\|_2^{\frac{2p}{p-2}} = \frac{p-2}{2p}\alpha^{\frac{p}{p-2}}$$

and on the other hand

$$\sup_{0\le s\le 1} E(\gamma(s)) \ge E(w) = \frac{p-2}{2p}\|\nabla w\|_2^2,$$

where $w=\gamma(s_\gamma)$ as calculated in (a). It further holds

$$\begin{aligned}\alpha &\le \|\frac{\nabla w}{\|w\|_p}\|_2^2 = \frac{\|\nabla w\|_2^2}{\|w\|_p^2}\\ &= \frac{\|\nabla w\|_2^2}{\|\nabla w\|_2^{\frac{4}{p}}} = \|\nabla w\|^{\frac{2p-4}{p}}.\end{aligned}$$

Combining these (in)equalities leads to (1).

**(d)** Let $\bar u$ be the element considered in (c). Then it holds

$$\beta = \sup E(\lambda\bar u) = \frac{2-p}{2p}\|\nabla \bar u\|_2^{\frac{2p}{p-2}} = \frac{2-p}{2p}\alpha^{\frac{p}{p-2}}, \tag{2}$$

which connects $\alpha$ and $\beta$. Let now $\tilde u = \frac{u}{\|u\|_p}$. As $dE(u)=0$, we have in particular $\langle dE(u),u\rangle = 0$, i.e.

$$\|u\|_p^p = \|\nabla u\|_2^2 = \frac{2p}{p-2}E(u)$$

as in (c). Therefore

$$\begin{aligned}\|\nabla\tilde u\|_2^2 &= \frac{1}{\|\tilde u\|_p^2}\|\nabla\tilde u\|_2^2 = \Big(\frac{2p}{p-2}E(u)\Big)^{1-\frac{2}{p}}\\ &= \Big(\frac{2p}{p-2}\beta\Big)^{\frac{p-2}{p}} = \alpha,\end{aligned}$$

where the last equality follows from (2).

### 2. Cerami.

**(a)** At each $u\in\tilde X$ we can find a vector satisfying the conditions. (This can be seen by taking a unit vector $v$ almost maximising $\langle dE(u),v\rangle$ and then normalising it to have norm almost $1+\|u\|$.) By the strict inequalities, the same vector will satisfy the conditions for all $v\in U_u$, where $U_u$ is a small neighbourhood of $u$. Thus we can cover $\tilde X$ by sets $U_u$. Then take a locally finite refinement of this cover and a Lipschitz continuous partition of unity subordinate to the refinement as in the lecture to get a global vectorfield, which is locally Lipschitz continuous.

Note that we can take the pseudo gradient vector field constructed in the lecture and multiply it with $(1+\|u\|_X)$ at each $u\in X$ to get exactly the desired vector field for this exercise.

**(b)** Given $\bar{\varepsilon} > 0$, let $0 < \varepsilon < \bar{\varepsilon}$ be such that $\varepsilon \leq \frac{1}{4}\delta$. Let $\tau\colon \mathbb{R} \to [0,1]$ be a smooth with $\tau(s) = 1$ if $|s-\beta| < \varepsilon$ and $\tau(s) = 0$ if $|s-\beta| > 2\varepsilon$. Let $e\colon X \to X$ be defined by

$$e(u) = \begin{cases} -\tau\Big(E(u)\Big)\,\tilde{e}^C(u), & \text{if } dE(u) \neq 0, \text{ i.e. } u \in \tilde{X}, \\ 0, & \text{if } dE(u) = 0. \end{cases}$$

Local Lipschitz continuity in $\tilde{X}$ is inherited from the vector field $\tilde{e}^C$ given in (a). Whenever $dE(u) = 0$, the assumption $N_{\beta,\delta} = \emptyset$ implies $|E(u) - \beta| \geq \delta \geq 4\varepsilon > 2\varepsilon$ such that $\tau \circ E$ and hence $e$ vanish in some neighbourhood of $u$ which is most Lipschitz. Therefore, the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\Phi(u,t) = e(\Phi(u,t)) \\ \quad \Phi(u,0) = u \end{cases}$$

has a solution $\Phi\colon X \times (-\tau,\tau) \to X$ at least for small $\tau > 0$. Moreover, given $t \in (-\tau,\tau)$

$$\begin{aligned} \|\Phi(u,t)\| - \|u\| \leq \|\Phi(u,t) - \Phi(u,0)\| &= \Big\|\int_0^t e(\Phi(u,t))\,ds\Big\| \\ &\leq \int_0^t \Big\|\tilde{e}^C(\Phi(u,t))\Big\|\,ds \leq \int_0^t 1 + \|\Phi(u,t)\|\,ds. \end{aligned}$$

By Gronwall's lemma $\|\Phi(u,t)\| \leq \Big(\|u\| + \tau\Big)e^{2t}$. Therefore, a solution $\Phi\colon X \times \mathbb{R} \to X$ exists.

**(c)** The map $t \mapsto E(\Phi(u,t))$ is non-increasing, since

$$\begin{aligned} \tfrac{d}{dt}E(\Phi(u,t)) &= \Big\langle dE(\Phi(u,t)), \tfrac{\partial}{\partial t}\Phi(u,t)\Big\rangle \\ &= -\tau\Big(E(\Phi(u,t))\Big)\Big\langle dE(\Phi(u,t)), \tilde{e}^C(\Phi(u,t))\Big\rangle \leq 0. \end{aligned}$$

To show the inclusion $\Phi(\mathsf{E}_{\beta+\varepsilon}, 1) \subseteq \mathsf{E}_{\beta-\varepsilon}$, consider $u \in \mathsf{E}_{\beta+\varepsilon} := \{u \in X \mid E(u) < \beta + \varepsilon\}$. Whenever $E(\Phi(u,t)) < \beta - \varepsilon$ for some $t \in (0,1)$, then also $E(\Phi(u,1)) < \beta - \varepsilon$, as $E(\Phi(u,\cdot))$ is non-increasing. Thus assume $E(\Phi(u,t)) \geq \beta - \varepsilon$ for all $t \in (0,1]$ towards a contradiction.

Clearly, also $E(\Phi(u,t)) \leq E(\Phi(u,0)) = E(u) < \beta + \varepsilon$ holds, such that $|E(\Phi(u,t)) - \beta| < \varepsilon < \delta$. Since $N_{\beta,\delta}$ is assumed to be empty, $\|dE(\Phi(u,t))\|_{X^*}\Big(1 + \|\Phi(u,t)\|\Big) \geq \delta$ follows necessarily. By definition, $\tau(\Phi(u,t)) = 1$ for all $t \in (0,1]$ and

$$\begin{aligned} \tfrac{d}{dt}E(\Phi(u,t)) &= -\Big\langle dE(\Phi(u,t)), \tilde{e}^C(\Phi(u,t))\Big\rangle \\ &< -\tfrac{1}{2}\|dE(\Phi(u,t))\|_{X^*}\Big(1 + \|\Phi(u,t)\|\Big) \leq -\tfrac{1}{2}\delta \leq 2\varepsilon. \end{aligned}$$

Consequently, $E(\Phi(u,1)) \leq E(u) - 2\varepsilon < \beta - \varepsilon$ in contradiction to the assumption. □