Solution 9

1. Präsenzaufgabe. First we show that $E \in C^1(H_0^1(\Omega))$. The $\int_\Omega |\nabla u|^2\, dx$-term is C^1. The calculations for the other term work similarly to Problem Set 1: We use the Theorem

Theorem. *Let $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. If the non-linear operator*

$$\begin{aligned} T\colon L^p(\Omega) &\to L^q(\Omega) \\ u &\mapsto g(\cdot, u(\cdot)) \end{aligned}$$

is well-defined, then it is also continuous.

We apply this Theorem to $T\colon L^p(\Omega) \to L^{\frac{p}{p-1}}$, $u \to g(\cdot, u)$, which is well defined by property 2 (and because Ω is a bounded domain) and then we get

$$\Big(\int_\Omega (g(x,u) - g(x,u_0))v\, dx\Big)^p \leq \|g(x,u) - g(x,u_0)\|_{\frac{p}{p-1}}^{p-1} \|v\|_p \to 0, \quad u \to u_0.$$

$dE(u)$ is given by

$$\begin{aligned} \langle dE(u), v\rangle_{H^{-1}\times H_0^1} &= \int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega g(x,u) v\, dx \\ &= \langle -\Delta u - g(\cdot, u), v\rangle_{H^{-1}\times H_0^1}. \end{aligned}$$

In the lecture we have seen that $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega) = (H_0^1(\Omega))^*$ is a linear isomorphism. We claim that $K\colon H_0^1(\Omega) \to H^{-1}(\Omega)$, given by $K(u) = g(\cdot, u)$ is a compact operator. Indeed, as $p < 2^*$, the inclusion $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact. By condition 2, T is continuous (nonlinear) $L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega)$. $L^{\frac{p}{p-1}}(\Omega)$ embeds into $H^{-1}(\Omega)$ (because $L^{\frac{p}{p-1}}(\Omega) = (L^p(\Omega))^*$). Therefore, K is compact as the composition of these maps.

$$\begin{tikzcd} H_0^1(\Omega) \arrow[r, "K", "\text{(non-linear)}"'] \arrow[d, hook, "\text{compact}"'] & H^{-1}(\Omega) & = (H_0^1(\Omega))^* \\ L^p(\Omega) \arrow[r, "T", "\text{continuous}"'] & L^{\frac{p}{p-1}}(\Omega) & = (L^p(\Omega))^* \arrow[u, hook] \end{tikzcd}$$

By Theorem 2.3.1. from the lecture it is therefore enough to show that every (P.-S.)$_\beta$-sequence is bounded, then we get the (P.-S.)$_\beta$-condition. Let (u_k) be a (P.-S.)$_\beta$-sequence, where $\beta \in \mathbb{R}$ is arbitrary.

As (u_k) is a (P.-S.)$_\beta$-sequence, we have

$$qE(u_k) - \langle dE(u_k), u_k\rangle_{H^{-1}\times H_0^1} = q(\beta + o(1)) + o(1)\|u_k\|_{H_0^1}, \quad k \to \infty.$$

On the other hand

$$\begin{aligned}
qE(u_k)-\langle dE(u_k),u_k\rangle &= q\Big(\frac{1}{2}\|\nabla u_k\|_2^2-\int_\Omega G(x,u_k)\,dx\Big)-\Big(\|\nabla u_k\|_2^2-\int_\Omega g(x,u_k)u_k\,dx\Big)\\
&=\frac{q-2}{2}\|\nabla u_k\|_2^2+\int_\Omega\left(g(x,u_k)u_k-qG(x,u_k)\right)\,dx\\
&\geq \frac{q-2}{2}\|u_k\|_{H_0^1}^2+\mathcal{L}^n(\Omega)\operatorname*{ess\,inf}_{x\in\Omega,v\in\mathbb{R}}\Big(g(x,v)v-qG(x,v)\Big),
\end{aligned}$$

when using the norm $\|u\|_{H_0^1}^2=\|\nabla u\|_2^2$. Combining these two we get

$$\frac{q-2}{2}\|u_k\|_{H_0^1}^2+o(1)\|u_k\|_{H_0^1}\leq q\beta+o(1)-\mathcal{L}^n(\Omega)\operatorname*{ess\,inf}_{x\in\Omega,v\in\mathbb{R}}\Big(g(x,v)v-qG(x,v)\Big),\quad k\to\infty.$$

By condition 3, the term $\Big(g(x,v)v-qG(x,v)\Big)$ is bounded from below by 0 for all $|v|\geq R_0$. For small $|v|$, condition 2 gives a lower bound in dependence of C, p and R_0. More precisely: $|g(x,u)|\leq C(1+R_0^{p-1})$ and therefore $G(x,u)\leq R_0C(1+R_0^{p-1})$ and then

$$\Big(g(x,v)v-qG(x,v)\Big)\geq-(CR_0+qR_0C)(1+R_0^{p-1}).$$

Therefore this term is bounded from below for all v and a.e. $x\in\Omega$ and we have shown that $\|u_k\|$ is bounded.

2. Boundary Value Problem. We will check the conditions for the mountain pass Lemma.

$E\in C^1(H_0^1(\Omega))$ and the (P.-S.)$_\beta$-condition for all $\beta\in\mathbb{R}$ were already checked in Exercise 1.

$E(0)=0$, because $G(x,0)=0$ for all x by definition of G.

We need to find some $\alpha,\rho>0$ satisfying: $\|u\|_{H_0^1}=\rho\Rightarrow E(u)\geq\alpha$. To bound $E(u)$ from below, we need to bound $G(x,s)$ from above. The idea is that for large s, $|g(x,s)|$ is bounded by a multiple of $|s|^{p-1}$ (condition 2) and for small s, $g(x,s)$ is bounded by εs for arbitrary small $\varepsilon>0$ (condition 1). To make this precise: Condition 1 tells us that given $\varepsilon>0$, there is $\delta(\varepsilon)>0$ such that $\frac{g(x,s)}{s}\leq\varepsilon$ for all $|s|<\delta(\varepsilon)$. (Note that we can only bound $g(x,s)$ from above, we cannot find a bound for $|g(x,s)|$.) Then

$$G(x,s)=\int_0^s g(x,t)\,dt=\int_0^s\frac{g(x,t)}{t}t\,dt\leq\int_0^s\varepsilon t\,dt=\frac{\varepsilon}{2}s^2,\quad |s|<\delta(\varepsilon).$$

For $|s|\geq\delta(\varepsilon)$, we use condition 2 which tells that we can find a constant $C(\varepsilon)$ (possibly much larger than the given constant C) such that $|g(x,s)|\leq C(\varepsilon)|s|^{p-1}$. So splitting the integral $\int_0^s g(x,t)\,dt$ in where $|s|<\delta(\varepsilon)$ and $|s|\geq\delta(\varepsilon)$ we can bound

$$G(x,s)\leq\frac{\varepsilon}{2}|s|^2+\frac{C(\varepsilon)}{p}|s|^p,$$

where this bound holds for all s. Therefore

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\varepsilon}{2}\int_\Omega |u|^2\,dx - \frac{C(\varepsilon)}{p}\int_\Omega |u|^p\,dx \\
&\geq \left(\frac{1}{2} - \frac{\varepsilon}{2\lambda_1} - \frac{C(\varepsilon)\tilde{C}}{p}\|u\|_{H_0^1}^{p-2}\right)\|u\|_{H_0^1}^2,
\end{aligned}$$

where $\lambda_1 = \inf\limits_{0\neq u\in C_c^\infty(\Omega)} \dfrac{\|\nabla u\|_2^2}{\|u\|_2^2} > 0$ is the first eigenvalue of $-\Delta$ as in Problem Set 6 and where we used the embedding $H_0^1 \hookrightarrow L^p$, i.e. $\|u\|_p^p \leq \tilde{C}\|u\|_{H_0^1}^p$. Taking first ε small enough (such that $\frac{1}{2} - \frac{\varepsilon}{2\lambda_1} > 0$) and then $\|u\|_{H_0^1} = \rho$ small enough, we get that

$$E(u) \geq \alpha := \left(\frac{1}{2} - \frac{\varepsilon}{2\lambda_1} - \frac{C(\varepsilon)\tilde{C}}{p}\rho^{p-2}\right)\rho^2 > 0.$$

Finally we need to find some u_1 with $E(u_1) < 0$. Therefore we will show that $E(\lambda u) \to -\infty$ $\lambda \to \infty$, if $u \neq 0$. So this time we need to bound $G(x,s)$ from below. One can check that condition 3 implies

$$s|s|^q \frac{d}{ds}\big(|s|^{-q}G(x,s)\big) \geq 0, \quad \text{if } |s| \geq R_0.$$

When we integrate this we get for $v \geq R_0$

$$\begin{aligned}
0 &\leq \int_{R_0}^v s|s|^q \frac{d}{ds}\big(|s|^{-q}G(x,s)\big)\,ds \leq |v|^{q+1}\int_{R_0}^v \frac{d}{ds}\big(|s|^{-q}G(x,s)\big)\,ds \\
&= |v|^{q+1}|s|^{-q}G(x,s)|_{s=R_0}^v = |v|G(x,v) - |v|^{q+1}R_0^{-q}G(x,R_0) \\
\Rightarrow\ & G(x,v) \geq c|v|^q,
\end{aligned}$$

where $c = R_0^{-q}\min\{G(x,R_0), G(x,R_0)\}$ and for $v < -R_0$ we integrate $\int_{-R_0}^v \ldots$ and get the same estimate. For $|v| \leq R_0$, we bound

$$|G(x,v)| \leq \operatorname*{ess\,sup}_{x\in\Omega,\,|v|\leq R_0} |G(x,v)| \leq CR_0(1 + R_0^{p-1}).$$

So inserting this into $E(\lambda u)$ we get

$$\begin{aligned}
E(\lambda u) &= \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,\lambda u)\,dx \\
&\leq C_1(u)\lambda^2 - \lambda^q\Big(\int_{\{\lambda|u|\geq R_0\}} c|u|^p\,dx\Big) + \mathcal{L}^n(\Omega)\operatorname*{ess\,sup}_{x\in\Omega,|v|\leq R_0}|G(x,v)| \\
&\to -\infty, \quad \lambda \to \infty,
\end{aligned}$$

where the convergence uses $q > 2$ and where $C_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$. Therefore for λ large enough $E(\lambda u) < 0$.

The mountain pass Lemma then gives a non-trivial solution $u \in H_0^1(\Omega)$. But we actually want to find solutions $u^+ \geq 0 \geq u^-$. Therefore we modify g as follows:

$$g^+(x,u) = \begin{cases} g(x,u) & u > 0 \\ 0 & u \leq 0 \end{cases}$$

$$g^-(x,u) = \begin{cases} 0 & u > 0 \\ g(x,u) & u \leq 0. \end{cases}$$

$g^\pm$ satisfy almost the same condtions as g, so we can apply the above calculations with modifications when we show $E(\lambda u) \to -\infty$: For g^+, the calculations in this step only work for $u \geq 0$ and for g^- they only work for $u \leq 0$. The rest of the above steps can be taken over to get nontrivial solutions $u^\pm$ of

$$\begin{cases} -\Delta u^+ = g^+(\cdot, u^+) & \text{in } \Omega, \\ \quad u^+ = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u^- = g^-(\cdot, u^-) & \text{in } \Omega, \\ \quad u^- = 0 & \text{on } \partial\Omega. \end{cases}$$

We need to check that $u^+ \geq 0 \geq u^-$ Then $g^+(x, u^+(x)) = g(x, u^+(x))$ and $g^-(x, u^-(x)) = g(x, u^-(x))$ for all x and therefore $u^\pm$ are solutions of the original equation.

We can show $u^+ \geq 0$ by writing $u^+ = v + w$, where v is the positive part of u^+ and w the negative part. Testing the PDE with w we then get

$$\|\nabla w\|_2^2 = \int_\Omega \nabla u^+ \nabla w \, dx = \int_\Omega g^+(x, u^+) w \, dx = 0,$$

because g^+ is 0 whenever w is non-zero. Therefore $w \equiv 0$.

Similarly, the positive part of u^- is 0.

3. Three Distinct Solutions.

(a) First note that E is coercive, because $\|u\|_4^4 \geq c\|u\|_2^4$ and therefore this term dominates the two negative terms.

As E is w.s.l.s.c. we can find minimisers of E in the weak closures of $M^\pm$. Note that for a weak limit u of elements in M^+ satisfies $\langle u, e_1 \rangle \geq 0$, so the weak closure is a subset of the norm closure. So we just need to show that these minima cannot lie in the set $N := \{u \in H_0^1(\Omega) \mid \langle u, e_1 \rangle_{H_0^1} = 0\} = \overline{\operatorname{span}\{e_2, e_3, \ldots\}}$, where e_i are the eigenfunctions of $-\Delta$

to the eigenvalue λ_i. To do this we estimate

$$\begin{aligned}
\inf_{u\in M^+} E(u) &\le \inf_{t>0} E(te_1),\\
\inf_{u\in M^-} E(u) &\le \inf_{t<0} E(te_1).\\
E(te_1) &= \frac{1}{2}\Big(\|\nabla te_1\|_2^2 - \lambda\|te_1\|_2^2\Big) + \frac{1}{4}\|te_1\|_4^4 - \int_\Omega te_1 f\,dx\\
&= \frac{1}{2}t^2\Big(\lambda_1-\lambda\Big)\|e_1\|_2^2 + \frac{1}{4}t^4\|e_1\|_4^4 - t\int_\Omega e_1 f\,dx\\
&\le \frac{1}{2}t^2\Big(\lambda_1-\lambda\Big) + \frac{1}{4}t^4\|e_1\|_4^4 + |t|\|e_1\|_2\|f\|_2 =: g(t).
\end{aligned}$$

As $\lambda > \lambda_1$, $g(t)$ is of the form $-A_1t^2 + A_2t^4 + A_3t$ with $A_1, A_2, A_3 > 0$. Whatever A_1 and A_2 are, we can always choose A_3 so small that g has a negative minimum for some $t > 0$. A_3 can be made small by letting $C(\lambda)$, the bound on $\|f\|_2$, be small. The above bound holds for all $t \in \mathbb{R}$, so both $\inf_{u\in M^+} E(u)$ and $\inf_{u\in M^-} E(u)$ ar bounded away from 0. We also note here that if $\|f\|_2$ gets smaller, then the bound on the infimum gets better, in the sense of more negative.

Now we need to bound $\inf_{u\in N} E(u)$ from below. Therefore we first note that for $u \in N$, which is a limit of linear combinations of $e_2, e_3, \ldots$, satisfies $\|\nabla u\|_2^2 \ge \lambda_2\|u\|_2^2$. (Because each e_i, $i > 1$ satisfies this inequality, then it holds for finite linear combinations and then also for limits.) Then we get

$$\begin{aligned}
E(u) &\ge \frac{1}{2}\Big(\lambda_2-\lambda\Big)\|u\|_2^2 + \frac{1}{4}\|u\|_4^4 - \int_\Omega uf\,dx\\
&\ge \frac{1}{2}\Big(\lambda_2-\lambda\Big)\|u\|_2^2 - \|u\|_2\|f\|_2.
\end{aligned}$$

It holds $\min_{t\in\mathbb{R}} \frac{1}{2}\Big(\lambda_2-\lambda\Big)t^2 - t\|f\| = -\frac{\|f\|_2^2}{2(\lambda_2-\lambda)}$ by differentiating w.r.t. t and therefore

$$\inf_{u\in N} E(u) \ge -\frac{\|f\|_2^2}{2(\lambda_1-\lambda)}$$

If $C(\lambda)$ gets smaller, then this lower bound gets arbitrarily close to 0, whereas the upper bound on $\inf E(te_1)$ is in some fixed distance away from 0 (and gets smaller for $C(\lambda)$ smaller), so we see that for $C(\lambda)$ small enough there holds

$$\inf E(te_1) < \inf_{u\in N} E(u) - \varepsilon$$

for some small $\varepsilon > 0$.

So we have shown that the minima of $\overline{M^\pm}$ need to be attained in the interior of these two sets, i.e. in $M^\pm$.

(b) In (a) we have already shown that $\inf_{u\in N} E(u) > E(u^+), E(u^-)$. If we show that $E \in C^1$ and satisfies (P.-S.)$_\beta$ for appropriate β, we can apply Thm. 2.3.2. which tells us that there is a third critical point.

We use similar estimates as in Exercise 1 to show C^1.

For (P.-S.)$_\beta$ we use coercivity. Because then every (P.-S.)$_\beta$-sequence is bounded. dE is of the form considered in Theorem 2.3.1. for $L = -\Delta$ and $K(u) = \lambda u - u^3 + f$. As a map $L^6 \to L^2$, $u \mapsto u - u^3 + f$ is continuous and bounded in the sense that $\|u - u^3 + f\|_2 \leq \mathcal{L}^n(\Omega)^{\frac{1}{3}} \|u\|_6^2 + \|u\|_6^3 + \|f\|_2$, i.e. if $\|u\|_6 \leq c$, there is a constant D such that $\|u - u^3 + f\|_2 \leq D$. The inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and therefore $(L^2(\Omega))^* \hookrightarrow (H_0^1(\Omega))^*$ is compact as well. Hence K is compact as the composition of a bounded and a compact map.

Then Thm. 2.3.2. tells us that there is a critical point u with $E(u) = \beta$, where

$$\beta = \inf_{\gamma \in \Gamma} \sup_{0 \leq s \leq 1} E(\gamma(s))$$
$$\Gamma = \{\gamma \in C^0([0,1]; H_0^1(\Omega)) \mid \gamma(0) = u^+,\ \gamma(1) = u^-\}.$$

As each of these paths γ has to pass by N (because $u^+ \in M^+$, $u^- \in M^-$), we get from the calculations in (a) that $\beta > E(u^+), E(u^-)$.