Differential Geometry
D-MATH, HS 2014

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Exercise Sheet 1

1. a) Prove that a closed curve γ in $\mathbb{R}^n$ has
$$\int_\gamma |k|\, ds \geqslant \frac{\pi}{2}.$$

   b*) [1]]Prove that a closed curve γ in $\mathbb{R}^2$ has
$$\int_\gamma |k|\, ds \geqslant 2\pi.$$

2. a) Compute the curvature k and torsion l at $t = 0$ for the curve
$$t \to (t, at^2, bt^3) \quad a, b \in \mathbb{R}$$

   b) Show that if a curve γ in $\mathbb{R}^3$ has identically vanishing torsion then γ lies in a plane.

   c) Suppose that a curve γ in $\mathbb{R}^3$ has a constant curvature and torsion. Show that γ must be a helix.

   d) Prove that any given smooth functions $k(s)$, $l(s)$, with $k(s) > 0$ determine a curve in $\mathbb{R}^3$ that is unique up to rigid motion of space (i.e. a composition of rotations and translations).

   Hint: First compute
$$\frac{d}{ds}\begin{pmatrix} \tau \\ N \\ B \end{pmatrix}$$
   in terms of
$$\begin{pmatrix} \tau \\ N \\ B \end{pmatrix}$$

[1]For $n = 3$ we a similar result.

Theorem 0.1. *Any closed curve γ in $\mathbb{R}^3$ has*
$$\int_\gamma |k|\, ds \geqslant 2\pi.$$
Moreover there is an equality if and only if γ is a plane convex curve.

Then apply the uniqueness and existence theorem for ODEs (you do not have to prove this!).

Theorem(*Existence and uniqueness for ODEs*[2])
Let $U \subseteq \mathbb{R}^n$ be an open set. Let $f_1, \dots f_n : U \to \mathbb{R}$ be Lipschitz continuous functions and let $x_0 := (x_0^1, x_0^2, \dots, x_0^n)$ be a point of U. For any $t_0 \in \mathbb{R}$ consider the ODE system

$$(*) = \begin{cases} \dot{y}^i(t) = f^i(y^1(t), \dots, y^n(t)) & i = 1, \dots n, \\ y^i(t_0) = x_0^i & i = 1, \dots n. \end{cases}$$

then

a) There exists a small open interval V containing t_0 and a continuously differentiable function $g(t) := (y^1(t), y^2(t), \dots, y^n(t)) : V \to \mathbb{R}^n$ that solves $(*)$.

b) Suppose that there are two solutions $g, \tilde{g}$ of $(*)$ defined on intervals V and $\tilde{V}$ respectively. Then the two functions agree on the intersection $V \cap \tilde{V}$.

3. *(For those new to topology)*

   a) Let X be a topological space. Show that if X is path connected, then X is connected.

   b) Let X, Y be a topological spaces. Show that if X is connected and the map $f : X \to Y$ is continuous, then $f(X)$ is connected.

   c) Let $U \subseteq \mathbb{R}^n$ be open set. Show that if U is connected, then it is path connected.

4. a) Let $M \subseteq \mathbb{R}^3$ be a surface, $p \in M$, $X \in T_pM$ an unit vector. Show that

   $$A_p(X, X) = \langle K_\gamma(p), N(p) \rangle,$$

   where γ is any curve in M satisfying $\gamma(0) = p$ and $\frac{d\gamma}{ds}(0) = X$ and K_γ is its curvature vector.

   b) Let M be a surface of revolution in $\mathbb{R}^3$ obtained by rotating a graph $y = f(x)$ about the x-axis. Compute the principle curvatures and principle directions of M.

Due on Wednesday October 1(resp. Friday October 3)

[2]For a proof see for examples C. Blatter, *Analysis* 11.26 www.math.ethz.ch/∼ blatter/