Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 10

1. (Killing fields on $\mathbb{R}^3$, continuation) Given $v \in \mathbb{R}^3$, recall the two vectorfields defined in exercise sheet 9, exercise 5

$$T_v(x) = v, \quad R_w(x) = w \times x, \quad x \in \mathbb{R}^3.$$

(a) Describe the flows $\Phi_t^{T_v}$, $\Phi_t^{R_v}$ geometrically.

(b) Determine by geometric reasoning conditions on v, w such that the flows $\Phi_t^{T_v}$, $\Phi_t^{R_w}$ commute. (We say that two diffeomorphisms ϕ, ψ *commute* if $\phi \circ \psi = \psi \circ \phi$.)

(c) Determine by computation conditions on v, w such that the Lie brackets $[T_v, R_w]$ vanishes.

2. Let X, Y, Z be smooth vector fields. Show that for any diffeomorphism ϕ,

$$\phi^*[Y, Z] = [\phi^* Y, \phi^* Z].$$

3. Let G be a Lie group, $e \in G$ the identity element. We call a vector field X on G *left-invariant* if $L_a^*(X) = X$ for all $a \in G$. Let $\mathcal{G} := \{\text{left invariant vector fields on } G\}$.

(a) For each $\tilde{X} \in T_e G$, there is a unique left-invariant vector field X with $X(e) = \tilde{X}$. (Thus $\mathcal{G}$ may be identified with the tangent space of G at the identity.)

(b) Prove that a left-invariant vector field is smooth.

(c) Prove that if X, Y are left-invariant, then so is $[X, Y]$. Prove that $\mathcal{G}$ forms a Lie subalgebra of the Lie algebra $C^\infty(TG)$. ($\mathcal{G}$ is called the *Lie algebra of* G. It is a remarkable fact that G can be reconstructed in a neighborhood of the identity from the finite, algebraic information contained in $\mathcal{G}$.)

4. (a) Let $GL(n, \mathbb{R})$ be the invertible $n \times n$ real matrices, with Lie algebra $\mathcal{GL}(n, \mathbb{R}) \cong T_{Id} GL(n, \mathbb{R}) = M^{n \times n}(\mathbb{R})$. Using the above exercise, for $A \in T_{Id} GL(n, \mathbb{R})$ find explicitly the unique left-invariant vector field X_A on $GL(n, \mathbb{R})$ such that

$$X_A\,(Id) = A.$$

(b) Show that the Lie bracket operation $[\cdot, \cdot]$ on $\mathcal{GL}(n, \mathbb{R})$ coincides (as a differential operator) with the anticommutator of matrices $AB - BA$, i.e.

$$[X_A, X_B] = X_{(AB - BA)}.$$

5. Compute the Lie algebras of $SO(3)$ and S^3 independently and compare.

Due on Wednesday December 10 (resp. Friday December 12)