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Exercise Sheet 11

1. Let X, Y, Z be smooth vector fields. The Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

expresses the diffeomorphism invariance of the Lie bracket in infinitesimal forms as follows. Let ϕ_t be the flow of X, and differentiate the identity

$$\phi_t^*[Y,Z] = [\phi_t^* Y, \phi_t^* Z].$$

at $t = 0$ (See exercise sheet 10, exercise 2.)

2. (a) Let $[X,Y] = 0$ and let ϕ_t, ψ_t be the flows of X and Y respectively. Prove

$$\phi_t \circ \psi_s = \psi_s \circ \phi_t,$$

wherever these are defined.

(b) Let $[X,Y] = 0$. Fix $p \in M$ and assume that $X(p), Y(p)$ are linearly independent. Prove there are coordinates $x^1, \ldots, x^n$ near p with

$$X = \frac{\partial}{\partial x^1}, \quad Y = \frac{\partial}{\partial x^2},$$

in a neighborhood of p.

(c*) Formulate and prove the analogous result for $X_1, \ldots X_n$ where $n = \dim M$.

3. Prove that the flows ϕ_s, ψ_t of the vector fields X, Y satisfy

(a) $\psi_t \circ \phi_s(x) = x + sX + tY + \frac{s^2}{2} D_X X + st D_Y X + \frac{t^2}{2} D_Y Y + O(|s|^3 + |t|^3)$,

(b) $\psi_{-t} \circ \phi_{-s} \circ \psi_t \circ \phi_s(x) = x + st[X,Y] + O(|s|^3 + |t|^3)$.

Note that (b), but not (a), has a meaning that is independent of the choice of coordinate system.

4. A car moves in the plane $\mathbb{R}^2$, identified with $\mathbb{C}$. The movement of the car is given by its position $(x_1(t), x_2(t)) \in \mathbb{R}^2$ and its direction given by the unit vector $e^{i\theta} \in S^1$. Moreover, we assume that the direction of movement always coincides with the main axis of the car. Now consider the vector fields $X(x_1, x_2, e^{i\theta}) := (\cos\theta, \sin\theta, ie^{i\theta})$ and $Y(x_1, x_2, e^{i\theta}) := (\cos\theta, \sin\theta, -ie^{i\theta})$ on the configuration space $M := \mathbb{R}^2 \times S^1$.

(a) Describe the geometric significance of the flows Γ_t^X, Γ_t^Y for driving around in $\mathbb{R}^2$.

(b) Compute $[X,Y]$.

(c) Why is parking so difficult? (Hint: see the formula in Exercise 3.)

Due on Wednesday December 17 (resp. Friday December 19)