Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 12

1. (a) Let $a(t) = a_X(t)$ be the integral curve of X starting at the identity. Show that $a(t)$ is a 1-parameter subgroup, i.e. $a(s+t) = a(s)a(t)$.

(b) Show that a is defined for all t. It follows that the flow of a left-invariant vector field is complete.

(c) Verify $a_{sX}(t) = a_X(st)$.

(d) The *exponential map (in the sense of Lie groups)*

$$\exp\colon T_eG \to G$$

is defined by

$$\exp(X) := a_X(1).$$

Show that exp is defined everywhere on T_eG and that

$$\exp(sX)\exp(tX) = \exp((s+t)X).$$

Warning: this does not always coincide with the exponential map of a Riemannian metric!

(e) Verify that for matrix groups, the above definition of $\exp(X)$ coincides with $\sum_{k=0}^{\infty} X^k/k!$ (see supplementary problem #16).

2. (a) Verify that for any real (or complex) square matrix A, $\det(\exp(A)) = \exp(\mathrm{tr}(A))$.

(b) Verify that $sl(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \mathrm{tr}(A) = 0\}$ is the Lie algebra $T_I SL(n,\mathbb{R})$ of $SL(n,\mathbb{R})$.

(c) Check that $SL(2,\mathbb{R})$ is diffeomorphic to the solid torus $S^1 \times B^2$, where B^2 is the open 2-dimensional disk.

(d) Show that the exponential map $\exp : sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is *not* surjective.

Sketch: First observe that the (complex) eigenvalues μ, μ^{-1} of a matrix $B \in SL(2,\mathbb{R})$ satisfy $\mu \in \mathbb{R}$ or $|\mu| = 1$. If they are real, then they are either both positive or both negative.

Claim: if the eigenvalues of A are both negative, then B cannot be written as $\exp(A)$ for any $A \in sl(2,\mathbb{R})$.

To prove this, calculate the form of $\exp(A)$ when $\mathrm{tr}(A) = 0$. By the Cayley-Hamilton theore, $A^2 = -\det(A)I$. Now split into cases according to the sign of det(A). If det$(A) = 0$, then $A^2 = 0$ and A is similar to the matrix $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$. If det$(A) < 0$, then the (complex) eigenvalues λ, $-\lambda$ of A satisfy $-\lambda^2 < 0$, so λ is real. If det$(A) > 0$, then λ, $-\lambda$ satisfy $-\lambda^2 > 0$, so $\lambda = i\theta$ is pure imaginary. In all three cases, we may compute the form of $\exp(A)$ explicitly, and we find that the case where both eigenvalues of $\exp(A)$ are negative is impossible.

(e) For comparison, recall that the exponential map of $SO(n)$ is surjective. (See Exercise Sheet 5 problem 2)

Happy Christmas!!!