Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 3

1. An isometry between surfaces in $\mathbb{R}^3$ preserves the Gauss curvature K but normally not k_1, k_2, H, or the principal directions of curvature (Serie 2 Exer. 2 was *quite* special in this respect.).

   a) Find a map from the plane to the cylinder that is locally an isometry (test it by seeing if it preserves the lenghts of curves).

   b) Show that the cone (minus the vertex) is locally isometric to the plane. Can it be realized by a single map?

   c) Compute K, k_1, k_2, H, of the cone and of the cylinder.

   d) A surface that is locally isometric to the plane is called a *developable surface*. Can you find other examples?

2. Let M be a set and let $\mathcal{A} := (U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ be a (smooth) atlas on M. Let $\mathcal{T}$ be the collection of "open sets" in M as defined in class:
$$W \in \mathcal{T} \Leftrightarrow \varphi_\alpha(W \cap U_\alpha) \text{ is open in } \mathbb{R}^n \text{ for all } \alpha \in \mathcal{A}.$$

   a) Prove $\mathcal{T}$ satisfies the three axioms of a topology.

   b) Recall that $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$. Prove that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism for each $\alpha \in \mathcal{A}$.

3. Let $\mathbb{RP}^n := \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$. For $p \neq 0$ in $\mathbb{R}^{n+1}$, let $[p]$ be the line through p and 0.

   a) Define a system of coordinates (an atlas) on $\mathbb{RP}^n$ that induces a smooth structure on $\mathbb{RP}^n$. (See Do Carmo, pp. 4, 20–22.)

   b) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ given by $F(x, y, z) := (x^2 - y^2, xy, xz, yz)$. Prove that F induces a well-defined map $f : \mathbb{RP}^2 \to \mathbb{R}^4$ characterized by $f([p]) := F(p)$ for any $p \in \mathbb{R}^3$, $||p|| = 1$.

   c) Prove that $f : \mathbb{RP}^2 \to \mathbb{R}^4$ is injective. (f is called the *Veronese embedding* of $\mathbb{RP}^2$ in $\mathbb{R}^4$, $\mathbb{RP}^4$ does not embed in $\mathbb{R}^3$).

   d) Express f in terms of the coordinate charts from (a) and observe that all coord. expressions of f are smooth (so *we call* f smooth).

Due on Wednesday October 15(resp. Friday October 18)