Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 4

1. (Topology practice) A topological space $(X, \mathcal{T})$ is called *compact* if every open cover of X contains a finite subcover, i.e. if for every family $\{U_\lambda\}_\lambda \subset \mathcal{T}$ with $X = \bigcup_\lambda U_\lambda$ there are finitely many indices $\lambda_1, \ldots, \lambda_n$ such that $X = \bigcup_{i=1}^{i=n} U_{\lambda_i}$.

   Prove the following properties, where X, Y are topological spaces:

   a) If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ then
   $$\mathcal{T}|_Y := \{U \cap Y \mid U \in \mathcal{T}\}$$
   is a topology on Y, called the the subspace topology (or the induced topology) on Y.

   b) If X is a subspace of Y, X is compact (with the induced topology), and Y is Hausdorff, then X is closed in Y.

   c) If X is closed in Y and Y is compact, then X (with the subspace topology) is compact.

   d) If $f\colon X \to Y$ is continuous and X is compact, then $f(X)$ is compact .

   e) If X, Y are compact, then $X \times Y$ is compact (with the product topology).

   f) A closed interval $[a, b] \subset \mathbb{R}$ is compact.

   g) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

2. A *topological sheaf* is a triple (f, X, Y) such that

   i) X, Y are topological space and $f : X \to Y$ is continuous,

   ii) f is surjective,

   iii) f is a local homeomorphism, i.e. for any $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism with respect to the induced topologies.

   Consider the following example. Let $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Set $Y_1 := \mathbb{R}$, $X_1 := \mathbb{R}_0 \cup \{0^+, 0^-\}$. Let $f_1 : X_1 \to Y_1$, defined by
   $$f_1(x) := \begin{cases} x, & \text{if } x \in \mathbb{R}_0, \\ 0 & \text{if } x \in \{0^+, 0^-\}\,. \end{cases}$$

Let $\mathcal{T}_{X_1}$ be the topology given by

$$\mathcal{T}_{X_1} = \{U \subseteq X_1 \mid f_1(U) \text{ is open in } Y_1\}$$

Observe that

a) f_1 is a topological sheaf and the maps $\phi_\pm := f_1|_{\mathbb{R}_0 \cup \{0^\pm\}}$ define a smooth atlas on X_1, but the induced topology is not Hausdorff.

b) More generally, any topological sheaf $f : X \to \mathbb{R}^n$ automatically acquires a smooth atlas consisting of its local homeomorphisms onto open subsets of $\mathbb{R}^n$.

c)* The sheaf of germs of holomorphic functions over $\mathbb{C}$ is Hausdorff and is a smooth manifold. The sheaf of germs of smooth real-valued functions over $\mathbb{R}$ is an extreme example of "non-Hausdorff manifold".

3. Consider $\mathbb{R}$ with its usual differentiable structure, induced by the chart $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, and consider also the differentiable structure induced by the chart $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x^3$.

   Show that the two differentiable structures are not equal, but that nevertheless the two differentiable manifolds thus defined are diffeomorphic.

4. (*Review of Quaternions*) Let Q denote the vector space $\mathbb{R}^4$ with basis $\{1, i, j, k\}$ and multiplication subject to the laws $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. (These make Q into an *algebra.*)

   a) Show that every non–zero element $V \in Q$ is invertible.

   b) Show that $|VW| = |V||W|$ for $V, W \in Q$.

   c) Show that $S^3 := \{V \in Q : |V| = 1\}$ has the structure of a group. (Hint: Set $u = a + bi + cj + dk$. It is useful to define $\bar{u} := a - bi - cj - dk$ and prove $\bar{u}u = |u|^2$.)

5. (Klein bottle) Let Γ be the group of isometries of $\mathbb{R}^2$ generated by the maps

$$(x, y) \mapsto (x + 1, -y)$$
$$(x, y) \mapsto (x, y + 1).$$

   a) Describe $\mathbb{R}^2/\Gamma$ geometrically.

   b) Give $\mathbb{R}^2/\Gamma$ a smooth structure in the "obvious way" and check that the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is smooth.

   b) How many pointwise linearly independent vector fields can **you** find on $\mathbb{R}^2/\Gamma$? (Hint: a vector field on $\mathbb{R}^2/\Gamma$ can be lifted up to one on $\mathbb{R}^2$ that is invariant under the action of the group Γ.)

Due on Wednesday October 22(resp. Friday October 25)