Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 5

1 (Quaternions) A *Lie group* is a smooth manifold endowed with a group structure such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

a) Show that $S^3 := \{V \in Q : |V| = 1\}$ is a Lie group.

b) Construct smooth vector fields X, Y, Z on S^3 such that $X(p)$, $Y(p)$, $Z(p)$ are independent for each p, and thereby prove $TS^3 \cong S^3 \times \mathbb{R}^3$.

Hint: Consider, for each $V \in S^3$, the left–multiplication map $L_V : S^3 \to S^3$, $W \mapsto VW$ and compute $(dL_V)_1 : T_1S^3 \to T_VS^3$ on the basis i, j and k.

2. Let $O(n) := \{A \in \mathbb{R}^{n\times n} \mid A^TA = I\}$ be the group of ortogonal matrices. (N.B. $\mathbb{K}^{p\times q}$ is the set of all $p \times q$ matrices with entries in $\mathbb{K}$.)

a) Compute a typical tangent vector to $O(n)$ at I as follows. Let $A(t)$ be a smooth curve in $\mathbb{R}^{n\times n}$ with $A(0) = I$, $A(t) \in O(n)$. Find an equation satisfied by $B := dA(0)/dt$.

b) Conversely, for any B satisfying the equation from a) find a curve $A(t)$ in $O(n)$ with initial velocity B. **Hint**: use the exponential map

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

c) A very beautiful picture of any such $A(t)$ comes by considering the diagonalization of orthogonal matrices to 2×2 blocks. Write $\mathbb{R}^n$ as the orthogonal sum of 2 dimensional subspaces V_i (and possible a 1-dimensional subspace W) and set each V_i rotating at constant angular speed θ_i.

d) What is the dimension of $O(n)$?

3. Recall from analysis that the function

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 \text{ if } x \leqslant 0, \end{cases}$$

is a smooth map. Prove: if U is an open set in a smooth manifold M and K is a compact subset of U, then there exists a *cutoff function for K in U*, i.e. a $\chi : M \to \mathbb{R}$ such that,

i) χ is smooth,

ii) $0 \leqslant \chi \leqslant 1$,

iii) $\operatorname{spt}(\chi) := \overline{\{p \in M \mid \chi(p) \neq 0\}}$ is a compact subset of U (we say $\operatorname{spt}(\chi)$ is *compactly contained in* U).

iv) $\chi \equiv 1$ on K.

4. Consider the following alternative version definition of a tangent vector: a tangent vector to M at p is a pair (p, Y) where Y is a *derivation at* p, meaning that Y is a linear map

$$Y \,:\, C^{\infty}(M) \to \mathbb{R}, \qquad u \mapsto Y \cdot u,$$

that satisfies the Leibniz rule at p:

$$Y \cdot (uv) = (Y \cdot u)\, v(p) + u(p)\, (Y \cdot v), \qquad u, v \in C^{\infty}(M).$$

a) It is easy to check that a tangent vector at p (as defined in class) is a derivation at p.

b) Prove that a derivation at p is a tangent vector at p (as defined in class).

Sketch of (b): Let Y be a derivation at p. We will show that Y may be expressed as a linear combination of $\left(\partial/\partial x^1\right)_{p,\psi}, \ldots, \left(\partial/\partial x^n\right)_{p,\psi}$.

i) Let $\psi = (\psi^1, \ldots, \psi^n) : U \to \mathbb{R}^n$ be a chart with $p \in U$. Let χ be a cutoff function for p in U that is constant in a neighborhood of p. For each $i = 1, \ldots, n$, define a special cut-off coordinate function on M by $\phi^i(x) := \chi(x)\psi^i(x)$ for $x \in U$, and extend ϕ^i by zero on the rest of M. Check that $\phi^i \in C^{\infty}(M)$.

ii) Define X in T_pM by

$$X := \sum_{i=1}^{n} X^i \left(\frac{\partial}{\partial x^i}\right)_{p,\psi}$$

where $X^i := Y \cdot \phi^i$. Prove: $Y \cdot \phi^i = X \cdot \phi^i$ for $i = 1, \ldots, n$.

iii) Prove that $Y \cdot u = X \cdot u$ for any u in $C^{\infty}(M)$, so Y belongs to T_pM. Hint: use a special version of the Taylor expansion with remainder to show that u may be written as $u(q) = u(p) + \sum_i a_i \phi^i(q) + \sum_i g_i(q)\phi^i(q)$, where a_i are constants and each g_i vanishes at p. Then use the fact that Y is a derivation at p. See Lee, *Introduction to Smooth Manifolds*, p. 64.

Due on Wednesday October 29 (resp. Friday October 31)