Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 6

1. a) Let $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ be a local diffeomorphism. Prove that its image is an open interval and that $f\colon \mathbb{R}^1 \to f(\mathbb{R}^1)$ is a diffeomorphism.

   b) Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is a local diffeomorphism, but is not a diffeomorphism onto its image.

   c) Show that a local diffeomorphism from a *compact* manifold to another manifold of the same dimension is a covering map.

2. Let $f\colon M \to N$ be a covering map and N connected.

   a) Show that the number k of elements in $f^{-1}(q)$ is constant on N. (We call f a k-sheeted covering).

   b) How many "different" 3-sheeted coverings can you find over S^1?

3. Let $\mathbb{CP}^n :=$ {complex lines in $\mathbb{C}^{n+1}$ through the origin}. For $z = (z^0, \dots, z^n) \neq 0$ in $\mathbb{C}^{n+1}$, define the complex line $[z] \in \mathbb{CP}^n$ by $[z] := \mathbb{C}\cdot z = \{\lambda z | \lambda \in \mathbb{C}\}$. It is conventional to write $[z] = [z_0 : \cdots : z_n]$, which emphasizes that only the ratios $z_i : z_j$ matter. We can endow $\mathbb{CP}^n$ with a smooth structure by declaring the functions

   $$I_j : \mathbb{C}^n \to \mathbb{CP}^n, \qquad I_j(w_1, \dots, w_n) := [w_1 : \cdots : w_{j-1} : 1 : w_j : \cdots : w_n],$$

   for $j = 0, \dots, n$, to be smooth parametrizations.

   a) Let $S^{2n+1} := \{z \in \mathbb{C}^{n+1} | \, |z| = 1\}$. Prove that the map $H : S^{2n+1} \to \mathbb{CP}^n$ given by $H(z) := [z]$ is a submersion. This map is called the *Hopf fibration*. The preimages $H^{-1}(q)$, $q \in \mathbb{CP}^n$, yield a decomposition of S^{2n+1} into circles (called *fibers*).

   b) Prove that every sphere of odd dimension carries at least one nowhere-vanishing vector field.

   c)* (For fun) Can S^2 be decomposed into a disjoint union of submanifolds diffeomorphic to S^1? Homeomorphic?

4. Let M be a smooth manifold.

   a) Let $p, q \in M$ and let $\gamma : [0, 1] \to M$ be a curve connecting p to q. Observe that any chosen orientation $E = E_\gamma(0)$ of T_pM propagates along γ to a unique path $E_\gamma(t)$ of orientations of $T_{\gamma(t)}M$ that is "continuous" in t (define this).

   b) Let γ be a closed curve in M, i.e. $\gamma(0) = \gamma(1)$. We say that γ is *orientation-preserving* if $E_\gamma(0)$ equals $E_\gamma(1)$ (for any choice of $E_\gamma(0)$); otherwise we say that

γ is *orientation-reversing*. Show that M is orientable if and only if every closed curve is orientation-preserving.

c) Show that the Möbius strip and the Klein bottle are not orientable.

d) Suppose M is orientable. How many orientations can it have?

5. Visualize the Hopf fibration $S^3 \to \mathbb{CP}^1 \cong S^2$ as follows.

a) Note that $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\} \cong S^2$ (the Riemann sphere).

b) Identify $\mathbb{C}^2$ with the quaternions Q by identifying $(z, w) = (a + bi, c + di)$ with $z + wj = a + bi + cj + dk$.

c) Identify $S^3 \backslash \{-1\}$ with $\mathbb{R}^3$ via stereographic projection from the point -1. Locate in the target $\mathbb{R}^3$ the images of the points $1, \pm i, \pm j, \pm k$ and the 6 "coordinate" great circles of S^3.

d) Partition S^3 into sets $T_r := \{(z, w) : |z| = \cos(r), |w| = \sin(r)\}$, $0 \leqslant r \leqslant \pi/2$. Observe that T_0 and $T_{\pi/2}$ are great circles of S^3 and are Hopf fibers. Observe that T_r, $0 < r < 1$ are all tori and are equidistant from each other (in the path metric of S^3) and that each T_r is a union of Hopf fibers. The middle torus $T_{\pi/4} = \{(e^{is}/\sqrt{2}, e^{it}/\sqrt{2}) : s, t \in \mathbb{R}\}$ is called the *Clifford torus*.

e) Visualize the Hopf fibration by drawing all of the Hopf fibers in $\mathbb{R}^3$ (after stereographic projection). The tori T_r are useful guides.

f) The quotient space $S^3/\sim$ is S^2, where $\sim$ is the equivalence relation whereby each Hopf fiber becomes a point. So S^3 is an S^2 of S^1's. Can you "see" the S^2 that is swept out as the fiber S^1 varies in S^3 (as represented via stereographic projection)?

g) Can you find the upper and lower hemispheres of the S^2 in your diagram?

Due on Wednesday November 5 (resp. Friday November 7)