Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 7

1. We call $J\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ a *complex structure* compatible with the Euclidean metric if $J^2 = -id$ and J is an isometry (w.r. to the euclidean metric). $(\mathbb{R}^{2N}, J)$ becomes a complex vector space isomorphic to $\mathbb{C}^N$. Note that J induces an orientation on $\mathbb{R}^{2N}$ via the real basis $e_1, Je_1, \ldots, e_N, Je_N$, where $e_1, \ldots, e_N$ is any complex basis of $(\mathbb{R}^{2N}, J)$. Let $\mathcal{J}(\mathbb{R}^{2N})$ be the set of all the complex structures compatible with the euclidean metric. Let $\mathcal{J}_0(\mathbb{R}^{2N})$ be the complex structures that induce the standard orientation and $\mathcal{J}_1(\mathbb{R}^{2N})$ be those that induce the opposite orientation. Show $\mathcal{J}_0(\mathbb{R}^4)$ and $\mathcal{J}_1(\mathbb{R}^4)$ are both diffeomorphic to S^2, so $\mathcal{J}(\mathbb{R}^4)$ is diffeomorphic to $S^2 \cup S^2$.

2) Let $\tilde{G}(n,k)$ be the space of *oriented* k-planes through 0 in $\mathbb{R}^n$. Show $\tilde{G}(4,2)$ is diffeomorphic to $\mathcal{J}_0(\mathbb{R}^4) \times \mathcal{J}_1(\mathbb{R}^4)$ and hence to $S^2 \times S^2$.

3. a) Show that a proper injective immersion is an embedding.

   b) Give an example of an injective immersion from $\mathbb{R}$ to a two-dimensional manifold whose image is dense in the target manifold and which is hence not an embedding.

4. Let M be a smooth manifold. Show TM is always orientable (even if M is not).

5. Let $\mathbb{Q} \equiv \mathbb{R}^4$ be the quaternions and write $\mathbb{R}^3 \equiv \{ai + bj + ck \in \mathbb{Q} | a, b, c \in \mathbb{R}\}$.

   a) Verify that the rule
   $$Ad_v\colon w \mapsto vwv^{-1}$$
   defines an action of S^3 on $\mathbb{R}^3$ by linear isometries (with respect to the usual inner product).

   b) Describe the action of an element $v = a + bi + cj + dk$ of S^3 on $\mathbb{R}^3$ *geometrically.*

   Hint: write $v = e^{tu}$ where $|u| = 1$, $u \in \mathbb{R}^3$, $t \in \mathbb{R}$. Note that $\left(e^{tu}\right)_{t\in\mathbb{R}}$ parametrizes a subgroup of S^3.

   c) Verify that the association $v \mapsto Ad_v$ gives a surjective homomorphism and a two-sheeted covering map from S^3 to $SO(3)$. Consequently, observe that $SO(3) \cong \mathbb{R}P^3$.

   d) Let US^2 denote the *unit tangent bundle* consisting of vectors in TS^2 with length 1 (in the usual metric). Identify US^2 with the set of all positively oriented orthonormal basis of $\mathbb{R}^3$, conclude that US^2 is diffeomorphic to $SO(3)$.

e) The composition

$$S^3 \xrightarrow{Ad} SO(3) \cong US^2 \xrightarrow{\pi} S^2 \cong \mathbb{C}P^1$$

is equivalent to the Hopf fibration.

Due on Wednesday November 12 (resp. Friday November 14)