Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 8

1. (a) Let G be a Lie group and K a discrete, normal subgroup of G. Show that the group homomorphism
$$G \to G/K$$
is a covering map. (It is called a *covering homomorphism* and G a *covering group* of G/K. If G is simply connected, we call G the *universal covering group* of G/K).

   (b) Show that a discrete normal subgroup of a connected Lie group G lies in the center of G.

   (c) Show that the universal covering group of $SO(4)$ is $S^3 \times S^3$ by finding a covering homomorphism
$$S^3 \times S^3 \to SO(4)$$
of degree 2.

   (d) Find all discrete normal subgroups of $S^3 \times S^3$ and the corresponding quotient groups.

   (e) Find all discrete normal subgroups of $SU(n)$.

2.* Verify the following:

   (a) Every subset of $\mathbb{R}^n$ is second countable.

   (b) Every closed subset of $\mathbb{R}^n$ is σ-compact.

   (c) Every smooth submanifold of $\mathbb{R}^n$ has a countable atlas.

   (d) For a smooth manifold M, the following are equivalent: (a) M is second-countable, (b) M is σ-compact, (c) M has a countable atlas.

   (e) If the smooth manifold M is second-countable then M is paracompact.

   (f) If the smooth manifold M is paracompact, then there is a (smooth, locally finite) partition of unity subordinate to any open cover of M.

   (g) Give an example of a paracompact smooth manifold that is not second-countable.

   Note: The Wikipedia article on paracompactness is useful, see also Lee, *Introduction to Smooth Manifolds.*

3. (a) What is the tangent space at the identity of $O(n)$? of $SO(n)$? of $U(n)$? of $SU(n)$? (Recall exercise sheet 3, exercise 3).

(b) Prove that $O(n)$ is a submanifold of $GL(n)$.

(c) Which of the above groups is connected? (Hint: diagonalize!)

4. (a) Show any closed set $A \subseteq \mathbb{R}^n$ is the zero set of some smooth function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

(b) Let $A \subseteq \mathbb{R}^n$ be closed. Show there exists open sets $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ such that ∂U_j is a smooth $(n-1)$-manifold and

$$A = \bigcap_{j=1}^{\infty} U_j.$$

(Hint: Sard's Theorem.)

(c) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that the set of critical values is $\mathbb{Q}$.

Due on Wednesday November 19 (resp. Friday November 21)