Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 9

1. Let X, Y be smooth vector fields and f, g smooth functions on a smooth manifold M. Prove

$$[fX, gY] = fg[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

2. Consider the ODE system

$$(\star) \quad \begin{cases} \frac{d}{dt}x(t) = X(x(t)), & t \in (0,T), \\ x(0) = x_0 \end{cases} .$$

Prove: if $X\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, then any two C^1 solutions $x(t), y(t) \in \mathbb{R}^n$ of $(\star)$ agree. Hint: Find a differential inequality for $|x(t) - y(t)|$.

3. Let $X \in C^k(TM)$ and let $\gamma\colon (-T,T) \to M$ be a C^1 integral curve of X. Show that γ is C^{k+1}.

4. Let $f : M \to N$ be smooth, $X \in C^\infty(TM), Y \in C^\infty(TN)$.

Define the *pushforward of* X *by* f via

$$f_*(X)(q) := df_{f^{-1}(q)}(X(f^{-1}(q)), \quad q \in Y$$

Define the *pullback of* Y *by* f via

$$f^*(Y)(p) := (df_p)^{-1}(Y(f(p))), \quad p \in X$$

(a) If f is bijective, $f_*(X)$ is well defined. If f is a diffeomorphism, $f_*(X) \in C^\infty(TN)$.

(b) If f is a local diffeomorphism, $f^*(Y)$ is well defined and lies in $C^\infty(TM)$.

(c) Suppose $f : M \to N$ and $g : N \to P$ are diffeomorphisms. Show

$$f^*g^* = (g \circ f)^*, \quad g_*f_* = (g \circ f)_*,$$

$$f^*f_* = id_{C^\infty(TM)}, \quad (f^{-1})^* = f_*.$$

5. (Killing fields on $\mathbb{R}^3$) Given $v \in \mathbb{R}^3$, define the vectorfields

$$T_v(x) := v, \quad R_v(x) := v \times x, \quad x \in \mathbb{R}^3.$$

(a) Compute $[T_v, T_w]$, $[T_v, R_w]$, and $[R_v, R_w]$ for $v, w \in \mathbb{R}^3$.

(b) Set $R_i := R_{\frac{\partial}{\partial x_i}}$ and $T_i := T_{\frac{\partial}{\partial x_i}}$ and write explicitly all the relations

$$[R_i, R_j], \quad [R_i, T_j], \quad [T_i, T_j]$$

for $i, j = 1, 2, 3$ in terms of the vectorfields $R_1, R_2, R_3, T_1, T_2, T_3$. Note that the system is closed.

Due on Wednesday December 3 (resp. Friday December 5)