Differential Geometry

D-MATH, HS 2014

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Exercise Sheet 9-10 DRAFT

NOTE: This week there is no assignment. Next week we will choose some of these exercises for the next exercise sheet.

1. Let X, Y be differentiable vector fields and f, g differentiable functions. Prove

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

2. Consider the ODE system

$$(\star) \quad \begin{cases} \frac{d}{dt}x(t) = X(x(t)), & t \in (0, T), \\ x(0) = x_0 \end{cases}.$$

Prove: if $X \colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, then any two C^1 solutions $x(t), y(t) \in \mathbb{R}^n$ of $(\star)$ agree. Hint: Find a differential inequality for $|x(t) - y(t)|$.

3. Let $X \in C^k(TM)$ and let $\gamma \colon (-T, T) \to M$ be a C^1 integral curve of X. Show that γ is C^{k+1}.

4. Let $f : M \to N$ be smooth, $X \in C^\infty(TM), Y \in C^\infty(TN)$.

Define the *pushforward of* X *by* f via

$$f_*(X)(q) := df_{f^{-1}(q)}(X(f^{-1}(q)), \quad q \in Y$$

Define the *pullback of* Y *by* f via

$$f^*(Y)(p) := (df_p)^{-1}(Y(f(p))), \quad p \in X$$

(a) If f is bijective, $f_*(X)$ is well defined. If f is a diffeomorphism, $f_*(X) \in C^\infty(TN)$.

(b) If f is a local diffeomorphism, $f^*(Y)$ is well defined and lies in $C^\infty(TM)$.

(c) Suppose $f : M \to N$ and $g : N \to P$ are diffeomorphisms. Show

$$f^* g^* = (g \circ f)^*, \quad g_* f_* = (g \circ f)_*,$$

$$f^* f_* = id_{C^\infty(TM)}, \quad (f^{-1})^* = f_*.$$

5. (Killing fields on $\mathbb{R}^3$) Given $v \in \mathbb{R}^3$, define the vectorfields

$$T_v(x) := v, \quad R_v(x) := v \times x, \quad x \in \mathbb{R}^3.$$

(a) Compute $[T_v, T_w]$, $[T_v, R_w]$, and $[R_v, R_w]$ for $v, w \in \mathbb{R}^3$.

(b) Write $R_i := R_{\frac{\partial}{\partial x_i}}$ and compute the relations

$$[R_i, R_j] = a_{ij}^k R_k, \quad i, j = 1, 2, 3.$$

(c) Describe the flows $\Phi_t^{T_v}$, $\Phi_t^{R_v}$ geometrically.

(d) Determine by geometric reasoning conditions on v, w such that the flows $\Phi_t^{T_v}$, $\Phi_t^{R_w}$ commute. (We say that two diffeomorphisms ϕ, ψ *commute* if $\phi \circ \psi = \psi \circ \phi$.)

(e) Determine by computation conditions on v, w such that the Lie brackets $[T_v, R_w]$ vanishes.

6. Let X, Y, Z be smooth vector fields. The Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

may be proven as follows:

(a) Show that for any diffeomorphism ϕ,

$$\phi^*[Y, Z] = [\phi^* Y, \phi^* Z].$$

(b) Take $\phi = \phi^t$ to be the local flow of X, and differentiate at $t = 0$.

7. Let G be a Lie group, $e \in G$ the identity element. We call a vector field X on G *left-invariant* if $L_a^*(X) = X$ for all $a \in G$. Let $\mathcal{G} := \{\text{left invariant vector fields on } G\}$.

(a) Prove that a left-invariant vector field is smooth.

(b) For each $\tilde{X} \in T_e G$, there is a unique left-invariant vector field X with $X(e) = \tilde{X}$. (Thus $\mathcal{G}$ may be identified with the tangent space of G at the identity.)

(c) Prove that if X, Y are left-invariant, then so is $[X, Y]$. Thus $\mathcal{G}$ forms a Lie subalgebra of the Lie algebra $C^\infty(TG)$. ($\mathcal{G}$ is called the *Lie algebra of* G. It is a remarkable fact that G can be reconstructed in a neighborhood of the identity from the finite, algebraic information contained in $\mathcal{G}$.)

8. (a) Let $GL(n, \mathbb{R})$ be the invertible $n \times n$ real matrices, with Lie algebra $\mathcal{GL}(n, \mathbb{R}) \cong T_{Id} GL(n, \mathbb{R}) = M^{n \times n}(\mathbb{R})$. Show that the Lie bracket operator $[\cdot, \cdot]$ on $\mathcal{GL}(n, \mathbb{R})$ coincides (as a differential operator) with the anticommutator of matrices $AB - BA$.

(b) Compute the Lie algebras of $SO(3)$ and S^3 independently and compare.

9. (a) Let $[X, Y] = 0$ and let ϕ_t, ψ_t be the flows of X and Y respectively. Prove

$$\phi_t \circ \psi_s = \psi_s \circ \phi_t,$$

wherever these are defined.

(b) Let $[X, Y] = 0$. Fix $p \in M$ and assume $X(p), Y(p)$ are linearly independent. Prove there are coordinates $x^1, \ldots, x^n$ near p with

$$X = \frac{\partial}{\partial x^1}, \quad Y = \frac{\partial}{\partial x^2},$$

on a neighborhood of p.

(c*) Formulate and prove the analogous result for $X_1, \dots X_n$ where $n = \dim M$.

10. Prove that the flows ϕ_s, ψ_t of the vector fields X, Y satisfy

$$\psi_{-t} \circ \phi_{-s} \circ \psi_t \circ \phi_s(x) = x + st[X, Y] + o(|s|^3 + |t|^3)$$

in any coordinate system.

11. A car moves in the plane $\mathbb{R}^2$, identified with $\mathbb{C}$. The movement of the car is given by its position $(x_1(t), x_2(t)) \in \mathbb{R}^2$ and its direction given by the unit vector $e^{i\theta} \in S^1$. Moreover, we assume that the direction of movement always coincides with the main axis of the car. Now consider the vector fields $X(x_1, x_2, e^{i\theta}) := (\cos\theta, \sin\theta, ie^{i\theta})$ and $Y(x_1, x_2, e^{i\theta}) := (\cos\theta, \sin\theta, -ie^{i\theta})$ on the configuration space $M := \mathbb{R}^2 \times S^1$.

(a) Describe the geometric significance of the flows Γ_t^X, Γ_t^Y for driving around in $\mathbb{R}^2$.

(b) Compute $[X, Y]$.

(c) Why is parking so difficult? (Hint: see the formula in Exercise 9.)

NO DUE DATE FOR THE MOMENT