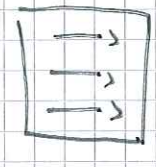
~~Exer~~ Exercise Sheet 10

1.

1. a) Note that easily we have

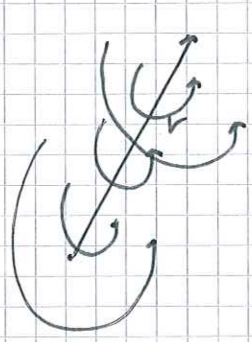
$$\Phi_t^{Tv}(x) = x + tv$$



Constant flow in direction v

$\Phi_t^{Rv}(x)$: Rotation along v with angle $t \cdot |v|$

See (*)



b) + c): Geometrically $\Phi_t^{Rv} \circ \Phi_s^{Rw} = \Phi_s^{Rw} \circ \Phi_t^{Tv}$ *

$\Rightarrow$ only when the flow of Rw don't "perturbe" the flow of $R\,Tv$

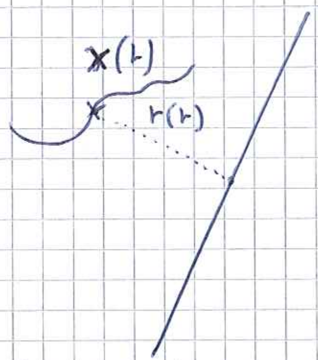
$\Leftarrow\Rightarrow$ $v \parallel \lambda w$

By computation: $[Tv, Rw](x) = v \times w = 0 \Leftrightarrow$ * holds (by the lecture)

$$\Rightarrow \underline{w = \lambda v \quad , \lambda \neq 0.}$$

(*) The picture * can be obtained via

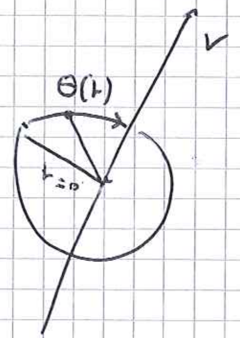
1) The axis fixed: $R_v(\lambda v) = 0$
2) Let $x(t)$ be the flow of R_v, then let $r(t)^2 = \left| x(t) - \frac{\langle x(t), v \rangle}{\langle v, v \rangle} v \right|^2$ be the distance from the axis (perpendicular!)



Claim. $\frac{d}{dt} r(t)^2 = 2 \langle \dot{x}(t) - \frac{\langle \dot{x}(t), v \rangle}{\langle v, v \rangle} v, x(t) - \frac{\langle x(t), v \rangle}{\langle v, v \rangle} \rangle$

$= 2 \langle v \times x(t) - \frac{\langle v \times x(t), v \rangle}{\langle v, v \rangle} v, x(t) - \frac{\langle x(t), v \rangle}{\langle v, v \rangle} v \rangle = 0$ $\blacksquare$

Since: $\frac{d}{dt} \langle x(t), v\rangle = \langle v \times x(t), v\rangle = 0$ we conclude



, where $\Theta: [-\varepsilon, \varepsilon) \longrightarrow [0, 2\pi]$

s.t. $\Theta(0) = 0$

i.e. we can work with cylindrical coord

$$e_1 = \frac{\langle x(t), v\rangle}{\langle v, v\rangle} v = \frac{\langle x, v\rangle}{\langle v, v\rangle} v$$

$$e_2 = r(0) \cdot \cos(\Theta(t)) \frac{(x - e_1)}{\|x - e_1\|}$$

$$e_3 = r(0) \sin(\Theta(t)) \frac{x \times v}{\|x \times v\|}$$

Since: $x(t) = \frac{\langle x, v\rangle}{|v|} \underbrace{\frac{v}{|v|}}_{e_1} + \cos(\Theta(t)) \overbrace{\left(\frac{x - \lambda e_1}{r(0)}\right)}^{e_2} r(0)$

$$+ \sin(\Theta(t)) \, r(0) \underbrace{e_1 \times e_2}_{e_3}$$

Conclusion From the equation $\dot{x}(t) = v \times x(t)$

$\Longrightarrow$ (with an easy calculation) $\Theta(t) = |v| t$

(2) pf:

The proof is an explicit calculation, we work in local coordinates
Let $X, Y \in C^\infty(M)$, WLOG we can write

$$Y = \sum Y^i \frac{\partial}{\partial x^i}, \quad X = \sum X^i \frac{\partial}{\partial x^i} \qquad Y^i, X^i \in C^\infty(U) \qquad U \subseteq \mathbb{R}^n$$

Recall that:

1) $$([Y,X](p))^i = \sum_k \left[\frac{\partial Y^i}{\partial x^k}(p) X^k(p) - \frac{\partial X^i}{\partial x^k}(p) Y^k(p) \right]$$

"same expression but permutated X with Y" $(Y \leftrightarrow X)$

2) $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeom, set $g := f^{-1}$, then

$$(((f^* Y)(p))^i := \sum_k \frac{\partial g^i(f(p))}{\partial x^k} \cdot Y^k(f(p))$$

Thm: $$([f^*(Y), f^*(X)](p))^i = \sum_k \frac{\partial [(f^* Y)(p)]^i}{\partial x^k} (f^*(X))^k(p) - \underbrace{\frac{\partial (f^*(X))^i(p)}{\partial x^k} (f^*(Y))^k(p)}_{=: (Y \leftrightarrow X)}$$

$$= \sum_k \left(\frac{\partial}{\partial x^k} \left(\sum_l \frac{\partial g^i(f(p))}{\partial x^l} \cdot Y^l(f(p)) \right) \right) \left(\sum_m \frac{\partial g^k}{\partial x_m}(f(p)) \cdot X^m(f(p)) \right) - (Y \leftrightarrow X)$$

Leibniz

$$\stackrel{\downarrow}{=} \sum_k \sum_l \left[\frac{\partial}{\partial x^k} \left(\frac{\partial}{\partial x^l} g^i(f(p)) \right) Y^l(f(p)) + \frac{\partial}{\partial x^l} g^i(f(p)) \cdot \frac{\partial}{\partial x^k} Y^l(f(p)) \right] \left(\sum_m \frac{\partial}{\partial x^m} g^k(f(p)) X^m(f(p)) \right)$$

$$- (Y \leftrightarrow X)$$

chain rule

$$\stackrel{\downarrow}{=} \sum_k \sum_l \left[\left(\sum_n \frac{\partial}{\partial x^n} \frac{\partial}{\partial x^l} g^i(f(p)) \cdot \frac{\partial f^n}{\partial x^k}(p) Y^l(f(p)) \right) + \frac{\partial}{\partial x^l} g^i(f(p)) \cdot \left(\sum_n \frac{\partial Y^l(f(p))}{\partial x^n} \frac{\partial f^n}{\partial x^k}(p) \right) \right]$$

$$\cdot \left(\sum_m \frac{\partial}{\partial x^m} g^k(f(p)) X^m(f(p)) \right) + (Y \leftrightarrow \cancel{X})$$

$$= \sum_{k,l,m,n} \left(\frac{\partial^2}{\partial x^n \partial x^l} g^i(f(p)) \cdot \frac{\partial f^n}{\partial x^k}(p) Y^l(f(p)) + \frac{\partial}{\partial x^l} g^i(f(p)) \frac{\partial Y^l}{\partial x^n}(f(p)) \cdot \frac{\partial f^n}{\partial x^k}(p) \right)$$

$$\cdot \frac{\partial}{\partial x^m} g^k(f(p)) X^m(f(p)) + (Y \leftrightarrow X)$$

Since $g = f^{-1}$ we have $\frac{\partial}{\partial x^k} f^n(p) \frac{\partial g^k}{\partial x^m}(f(p)) = \delta_{mn}$

Thus: we obtain

$$\sum_{k,l,m,n} \frac{\partial^2 g^j}{\partial x^m \partial x^l}(f(p))\, Y^l(f(p))\, X^m(f(p)) + \frac{\partial g^j}{\partial x^l}(f(p)) \frac{\partial Y^l}{\partial x^m}(f(p))\, X^m(f(p))$$

$$- \frac{\partial^2 g^j}{\partial x^m \partial x^l}(f(p))\, Y^l(f(p))\, X^m(f(p)) - \frac{\partial g^j}{\partial x^l}(f(p)) \frac{\partial Z^l}{\partial x^m}(f(p))\, X^m(f(p))$$

$$= \left(f^* [Y,X](p) \right)^j \quad \boxed{}$$

~~Other ways (Thanks to ...)~~

~~$\psi: M \to N$ diffeom, then $\psi^*[X,Y] = [\psi^* X, \psi^* Y]$ $X, Y \in C^\infty(TM)$~~

~~pf: $f \in C^\infty(M)$ $((\psi^* X) f)(p) = (\psi^* X)(p) f = [d\psi ...]_{\psi(p)}$~~

3) Let G be a Lie group, recall that for $a \in G$ we define

$$L_a : G \longrightarrow G$$
$$b \longmapsto a.b$$

$X \in C^\infty(TG)$ is called left invariant if $L_a^*(X) = X \quad \forall a \in G$

a) Let $\tilde{G} := \{ X \in C^\infty(TG) \mid X \text{ left invariant} \}$

Show: for each $\tilde{Y} \in T_e G$ there exists an unique vector field $Y \in \tilde{G}$ s.t. $Y(e) = \tilde{Y}$

pf: Let $Y \in \tilde{G}$ then $L_a^* Y(p) \overset{!}{=} Y(p) \quad \forall a \in G$

then $L_g^*(Y)(p) = (dL_g)_p^{-1} \; Y(gp) \overset{!}{=} Y(p)$

$\Rightarrow Y(gp) = (dL_g)_p \; Y(p)$

Thus: $Y(g) = (dL_g)_e \, Y(e)$

Consider the map $h : \tilde{G} \longrightarrow T_e G$
$\qquad Y \longmapsto Y(e)$

by above for $\tilde{Y} \in T_e G$ set $Y(g) := (dL_g)_e \tilde{Y}$, then

$$L_a^* Y(g) = (dL_a)_g^{-1} \, Y(a.g) = (dL_a)_g^{-1} (dL_{a.g})_e \tilde{Y} = (dL_g)_e \tilde{Y}$$

$\Rightarrow h$ is surj (so we have 3b), the injectivity:

$$h(Y_1) = h(Y_2) \Longleftrightarrow \underbrace{(dL_g)_e Y_1 = (dL_g)_e Y_2}_{\text{isom}} \Longleftrightarrow Y_1 = Y_2$$

$\boxtimes$ 3.a)

3.b) $X \in \tilde{G} \Rightarrow X \in C^\infty(TG)$

pf: by above $X \in \tilde{G}$ then $X(g) := (dL_g)_e^{-1} \tilde{X}$, $\tilde{X} \in T_e G$.

We recall some important fact

1) $f : M \to N \quad C^\infty \Rightarrow df : TM \to TN$ is C^∞
$\qquad (p, X) \longmapsto (f(p), df_p X)$

2) if M smooth manifold, then
$\pi : TM \to M$, $(p, X) \longmapsto p$, $\quad i : M \to TM$, $p \longmapsto (p, 0)$ are smooth.

Now: Consider the map (multiplication of Lie groups)

$$\mu : G \times G \longrightarrow G$$
$$(a, b) \longrightarrow a \cdot b$$

Thus:

$$\begin{array}{ccc} d\mu : T(G \times G) & \longrightarrow & TG \\ \| & & \\ TG \times TG & & \\ \uparrow f & & \\ G \times T_e G & & \end{array}$$

where $f : G \times T_e G \longrightarrow TG \times TG$ is smooth since $T_e G \longrightarrow TG$ is smooth
$(a, X) \longrightarrow ((a, 0), (e, X))$

$\Rightarrow d\mu \circ f$ is smooth

Claim: $d\mu \circ f = (dL_{(\cdot)})_e$

pf: let $(a, X) \in G \times T_e G$, then $d\mu\, f(a, X) = d\mu((a, 0), (e, X))$

- let $\gamma_1 : (-\varepsilon, \varepsilon) \to G$ s.t. $\gamma_1(t) = a \quad \forall t \in (-\varepsilon, \varepsilon)$, then

$\dot\gamma_1(0) = 0$, $\gamma_1(0) = a$

- let $\gamma_2 : (-\varepsilon, \varepsilon) \to G$ s.t. $\gamma_2(0) = e$, $\dot\gamma_2(0) = X$

Note that $\Gamma(t) := (\gamma_1(t), \gamma_2(t))$ is a path in $G \times G$ s.t.
$\Gamma(0) = (a, e)$, $\dot\Gamma(0) = (0, X)$ then

$$d\mu((a, 0), (e, X)) = d\mu_{(a,e)}(0, X) = \frac{d}{dt}\Big|_{t=0} \mu(\Gamma(t)) =$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_1(t) \cdot \gamma_2(t) = \frac{d}{dt}\Big|_{t=0} L_a(\gamma_2(t))$$

$$= (dL_a)_e X$$

$\blacksquare$ 3b)

3.b) Let M be a manifold (smooth), then $C^\infty(TM)$ is a vector space

The proof is easy:

Let $X, Y \in C^\infty(TM)$ for a given chart $\psi: U \to \mathbb{R}^n$ we can write

$$X = \sum X^i_u \left(\frac{\partial}{\partial x^i}\right)_\psi, \quad Y = \sum Y^i_u \left(\frac{\partial}{\partial x^i}\right)_\psi \in C^\infty(TU)$$

where $X^j_u, Y^j_u \in C^\infty(U)$. We define $Z = \lambda X + \mu Y$, $\lambda, \mu \in \mathbb{R}$ via

For any charts $\phi: V \to \mathbb{R}^n$

$$Z = \sum \lambda X^i_V \left(\frac{\partial}{\partial x^i}\right)_\phi + \mu Y^i_V \left(\frac{\partial}{\partial x^i}\right)_\phi$$

it is not difficult check that Z is well defined and that it is in $C^\infty(TM)$.

<u>Other immediate proof</u>

Do you remember that $X_p \in T_pM$ is equivalent to a map

$$X_p: C^\infty(M) \to \mathbb{R}$$

<u>s.t.:</u> $$X_p(f \cdot g) = X_p(f)\, g(p) + f(p)\, X_p(g) \quad ?$$

well there is a similar story with vectorfields:

<u>Additional Exercise</u> Show that the two definition are equivalent

<u>Def 1</u>: $X \in C^\infty(TM) \iff X: M \to TM$ smooth and $\pi \circ X = id_M$

<u>Def 2</u>: $X \in C^\infty(TM) \iff X$ defines a map $X: C^\infty(M) \to C^\infty(M)$

$$f \longmapsto X(f)$$

s.t $$X(fg) = X(f)g + f X(g)$$

<u>pf</u>: See Lee "intro to C^∞ mfld" page 86

<u>Corollary:</u> $C^\infty(TM)$ is a vector space

<u>pf</u>: <u>use def 2</u> $\to$ <u>Immediate!</u>

<u>Rmk:</u> What is the dimension of $C^\infty(TM)$?

<u>Ex:</u> $M = \mathbb{R}$ $\quad X: M \to TM$ is $X = h(x) \cdot \frac{\partial}{\partial x}$, thus $C^\infty(T\mathbb{R}) = \underline{\underline{C^\infty(\mathbb{R})}}$

and $\underline{\dim C^\infty(\mathbb{R}) = \infty}$! In general $C^\infty(TM) = \underline{\infty\text{-dim v.space}}$

· If you have a global frame $X_1, \dots, X_n \in C^\infty(TM)$ then any $X \in C^\infty(TM)$ may be written as

$$X = f_1 \cdot X_1 + \dots + f_n X_n$$

$f_i \in C^\infty(M)$. Of course in general the f_i are not constant. Thus

· A frame is not a basis in sense of vector spaces.

· if you evaluate at $p \in M$

$$X(p) = f_1(p) X_1(p) + \dots + f_n(p) X_n(p) \quad \in T_pM$$

Conclusion

A frame gives at each point a base of T_pM $\neq$ basis of $C^\infty(TM)$ (over $\mathbb{R}$)

Question: Over $C^\infty(M)$?

Now: The exercise

Claim: $\tilde{\mathcal{G}}$ is a vector space

pf: $X, Y \in \tilde{\mathcal{G}}$ then $L_a^*(\lambda X + \mu Y) = L_a^*(\lambda X) + L_a^*(\mu Y) = \lambda X + \mu Y$ ☐

Claim: $[-,-]: \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$

pf: $L_a^*[X,Y] \underset{\text{ex 2}}{\overset{\uparrow}{=}} [L_a^* X, L_a^* Y] = [X,Y]$

Conclusion: 1) M smooth manifold, $C^\infty(TM)$ vector space, $[-,-]: C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ is bilinear, antisymm and satisfies the Jacobi identity
$\Rightarrow (C^\infty(TM), [-,-])$ is a Lie algebra. (∞-dimensional)

2) $M = G$ Lie group, then $[\tilde{\mathcal{G}}, \tilde{\mathcal{G}}] \subseteq \tilde{\mathcal{G}}$
$\Rightarrow (\tilde{\mathcal{G}}, [-,-])$ is a subalgebra of $C^\infty(TG)$. It is finite dimensional since $\tilde{\mathcal{G}} \cong T_eG$

3) Advanced fact From $\tilde{\mathcal{G}}$ it is possible rebuild all G !

Idea: $\exp: \tilde{\mathcal{G}} \longrightarrow G$
$\tilde{X} \longrightarrow \Phi_1^{\tilde{X}}$
called exponential map

4) Recall the computations from exercise 2, Exercise sheet 8 (*)

a) Recall that $M^{n\times n} = \{ n\times n \text{ matrix with real entries}\} \simeq \mathbb{R}^{n^2}$
and that $GL(n)$ is an open set of $\mathbb{R}^{n^2}$ via

$$GL(n) = \underset{\substack{\uparrow\\ \text{continuous}\\ \text{map}}}{\det}{}^{-1}\Big(\underset{\substack{\uparrow\\ \text{open}}}{(-\infty,0)} \cup \underset{\substack{\uparrow\\ \text{open}}}{(0,\infty)} \Big)$$

$\Rightarrow$ $GL(n)$ is n^2 submanifold of $M^{n\times n}$, thus (since $M^{n\times n} \simeq \mathbb{R}^{n^2}$) we have

$$T_{id}\, GL(n) \simeq M^{n\times n}$$

Moreover $GL(n)$ is a Lie group (see (*)).
Denote with $\mathfrak{gl}(n)$ the Lie algebra of $GL(n)$ (see exercise 3)
By exercise 3.a) we can compute explicitly an element of $\mathfrak{gl}(n)$, i.e for each $X_A \in \mathfrak{gl}(n)$ there exists an $A \in T_{id}\, GL(n) = M^{n\times n}$ s.t

$$X_A : GL(n) \longrightarrow T\, GL(n)$$
$$u \longrightarrow X_A(u)$$

can be written as $X_A(u) = (dL_u)_{id}\, A$ with $A = X(id)$.

Claim: $B, C \in M^{n\times n}$, $(dL_B)_{id}\, C = BC$

pf: let $\gamma : (-\varepsilon,\varepsilon) \longrightarrow M^{n\times n}$ be a path with $\dot\gamma(0) = C$ (componentwise!)
then $\gamma(0) = id$

$$\big((dL_B)_{id} C\big) = \frac{d}{dt}\Big|_{t=0} L_B(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} B\cdot\gamma(t) \underset{\substack{\uparrow\\ \text{multiplication}\\ \text{with scalar!}}}{=} B\cdot\dot\gamma(0) = B\cdot C$$

$\square$ Claim

Thm: $X_A : GL(n) \longrightarrow TGL(n)$
$B \longmapsto (B, B\cdot A)$

b) By exercise 3 we have

$\mathfrak{gl}(n)$ sub lie algebra of $C^\infty(TGL(n))$, i.e.

$$[\mathfrak{gl}(n), \mathfrak{gl}(n)] \subseteq \mathfrak{gl}(n)$$

Let X_A, X_B defined as in a) with $A, B \in T_{id}GL(n)$.
Since $[X_A, X_B] \in \mathfrak{gl}(n)$ we have to check only what happen at the identity.
Now since the chart of $GL(n)$ is given by

$$\begin{array}{ccccc} GL(n) & \longrightarrow & \mathbb{R}^{n\times n} & \longrightarrow & \mathbb{R}^{n^2} \\ A & & & & a^{ij} \end{array}$$

For $A = (a^{ij})$ $B = (b^{ij}) \in \mathbb{R}^{n^2}$ we have

$$X_A(u) = u \cdot A \rightsquigarrow X_A(u) = \sum_{ij} X_A^{ij}(u) \frac{\partial}{\partial u^{ij}} = \sum_{ij} \sum_k (u^{ik} a^{kj}) \frac{\partial}{\partial u^{ij}}$$

$$X_B(u) = u \cdot B \rightsquigarrow X_B(u) = \dots = \sum_{ij} \sum_k (u^{ik} b^{kj}) \frac{\partial}{\partial u^{ij}}$$

Thus:

$$[X_A, X_B]^{ij}(u) = \sum_{lm} \left[\frac{\partial}{\partial u^{lm}} \left(\sum_k u^{ik} a^{kj} \right) \right] \cdot \left(\sum_k u^{lk} b^{km} \right) - (A \leftrightarrow B)$$

$$= [ABu - BAu]^{ij}$$

Since: $[X_A, X_B](id) = AB - BA$ we have

$$[X_A, X_B] = X_{(AB-BA)}$$

In particular: $(\mathfrak{gl}(n), [-,-])$ is isomorphic (as a lie algebra) to $(M^{n\times n}, [-,-])$

where: $$\begin{array}{rccc} [-,-]: & M^{n\times n} \times M^{n\times n} & \longrightarrow & M^{n\times n}, \\ & (A, B) & \longrightarrow & AB - BA \end{array}$$ (usual commutator)

5) SO(3)

Recall (from ex. sheet 8, ex 2 or ex. sheet 7, ex 5) that

$$T_{id}\, SO(3) = \{ U \in \mathbb{R}^{3\times 3} \mid U^T + U = 0 \}$$

Idea: Find $T_{id}\, O(3)$ and show that $SO(3)$ is a connected component of $O(3)$.
in particular its dimension is 3.

Consider the map

$$i : SO(3) \hookrightarrow GL(3)$$

and denote with $i(SO(3)) \subset GL(3)$. Note that it is a submanifold of $GL(n)$.
and since $id \in i(SO(3)) \cap GL(3)$ we have

$$T_{id}\, i(SO(3)) \subseteq T_{id}\, GL(3)$$

Claim: let A, B s.t. $A^T = -A$, $B^T = -B$, then $(AB-BA)^T = -(AB-BA)$

pf: $(AB-BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB$ ☒

Now: • $i(SO(3))$ is a Lie group, let $\widetilde{i(SO(3))}$ be its Lie algebra
• by exercises 3,4 we have

$$X \in \widetilde{i(SO(3))} \quad \text{then } \exists\ A \in T_{id}\, i(SO(3)) \subset T_{id}\, GL(3)$$

s.t.

$$X_A(U) = U \cdot A \quad , \forall\, U \in i(SO(3))$$

Claim: $\widetilde{i(SO(3))}$ is a sub Lie algebra of $\mathfrak{gl}(n)$

pf: Clearly $\widetilde{i(SO(3))}$ is a subvector space of $\mathfrak{gl}(n)$, now let
$X_A, X_B \in \widetilde{i(SO(3))}$, $A, B \in T_{id}\, i(SO(3))$

$$[X_A, X_B] \underset{\substack{\uparrow \\ ex\, 4}}{=} X_{AB-BA} \quad , \text{since } AB-BA \in T_{id}\,(SO(3)) \Rightarrow [\widetilde{i(SO(3))}, \widetilde{i(SO(3))}] \subseteq \widetilde{i(SO(3))}$$

Conclusion: $i : SO(3) \hookrightarrow i(SO(3))$ is equal to the identity because
$SO(3) \subset GL(n)$ by def. ☒ Claim

Fact:
let G, H be Lie groups, $f : G \to H$ be a smooth group homomorphism,
then f induces a Lie algebra homomorphism

$$f_* : \widetilde{G} \to \widetilde{H} \quad \left(\text{obtained from: } (df)_e : T_e G \to T_e H\right)$$

where $f_*(X_v) = X_{(df)_{id}(v)}$ (check ☺)

Rmk: • Read Lee, "intro to C^∞ mfld", page 55. (for more details ☺)
• He uses push forward, we use pullback but conclusion are the same!

Choosing the basis of $T_{id}\, SO(3)$ we have:

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$$

$$\Rightarrow A_1 \cdot A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow [A_1, A_2] = +A_3$$

$$\Rightarrow A_1 A_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \Rightarrow [A_1, A_3] = A_2$$

$$\Rightarrow A_2 A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow [A_2, A_3] = -A_1$$

S^3, Fact: since $Ad: S^3 \to SO(3)$ is a 2 sheeted cover we expected (by *) different lie algebra
BUT since the cover is universal, the lie algebra are the same!

Recall that $S^3 \subset Q$ $p \in S^3$ $p = (a + bi + ck + dj)$ s.t. $a^2+b^2+c^2+d^2=1$

$T_{id}\, S^3 = (i, k, j)$ obtained from the path $\gamma_x(t) = \cos(t) + x\sin(t)$, $x = i, j, k$

Claim: let $X \in \tilde{S}^3$, then there exists an $p \in T_{id} S^3$ s.t. $X: S^3 \to TS^3$ $X(q) = (q, q\cdot p)$

pf: by exercise 3, let γ be a path $\gamma: (-\varepsilon, \varepsilon) \to S^3$ s.t. $\gamma(0) = id$, $\dot\gamma(0) = p$

then $(dL_q)_e\, p = \frac{d}{dt}\Big|_{t=0} q\cdot\gamma(t) = q\cdot p$

Thus: we will write X_p for the element of $\tilde{S}^3$. ☒

Claim*: $[X_p, X_q] = X_{(pq-qp)}$ (see next page)

pf: Explicit calculation or using lie derivative (only for next week!) ☒

Conclusion: $(\tilde{S}^3, [\cdot,\cdot])$ is isomorphic as a lie algebra to $(R, [\cdot,\cdot])$

where: $R \times R \to R$
$(p, q) \mapsto pq - qp$

$R = \{ai + bj + ck \mid a, b, c \in \mathbb{R}\} = T_{id}\, S^3$

Other idea: Set $\mathbb{H} \setminus \{0\} = GL(1, \mathbb{H})$ (1×1 matrices...)
$\Rightarrow$ then we can use the same argument as above since $\mathbb{H}$ is a field!)

Relations $[i,j] = ij - ji = 2k$, $[i,k] = ik - ki = 2j$, $[j,k] = jk - kj = 2i$

Additional computation:

Consider the map $Ad: S^3 \to SO(3)$, then recall that:

1) The Lie algebra structure of $T_1 S^3$ is:
$T_1 S^3 = R = \{ai + bj + ck \mid a,b,c \in \mathbb{R}\}$, $[x,y] = xy - yx$

2) The Lie algebra structure of $SO(3)$ is given by
$T_{Id} SO(3) = so(3) = \{A \in M^{3\times 3} \mid A^T = -A\}$, $[A,B] = AB - BA$

Claim: Ad induces an iso of Lie algebras

Pf: $\mu_1 : (-\varepsilon, \varepsilon) \to S^3$, $\mu_1(t) = \cos(t/2) + i \sin\left(\frac{t}{2}\right)$, $\dot{\mu}_1(0) = i/2$

μ_2 " $\mu_2(t) = \cos(t/2) + j \sin(t/2)$ $\dot{\mu}_2(0) = j/2$

μ_3 " $\mu_3(t) = \cos(t/2) + k \sin(t/2)$ $\dot{\mu}_2(0) = k/2$

Now: By Ex Sheet 7, ex 5 c), we have (Attention, maybe there are some index-mistakes :-))

$$Ad_{\mu_1(t)} = e^{tA_1}, \quad Ad_{\mu_2(t)} = e^{tA_2}, \quad Ad_{\mu_3(t)} = e^{tA_3}$$

($\triangle$ the notation is different), then

$$(d\, Ad_e)\, i = \frac{d}{dt} e^{\hat{\mu}_1(t)} = 2A_1, \quad (d\, Ad_e)\, j = 2A_2, \quad (d\, Ad_e)\, k = 2A_3$$

$\Rightarrow$ Lie algebra isom. via $dAd_e[i,j] = [dAd_e i, dAd_e j] = [2A_1, 2A_2] = 4A_3$ ✓

Rmk: proof of claim* using Lie derivative (sketch) [For next week!]

Since: $[X,Y] = L_X Y$ we have:

Let $X_p : S^3 \to TS^3$, $q \mapsto (q, q\cdot p)$, $X_{p'} : S^3 \to TS^3$, $q \mapsto (q, q\cdot p')$

Thm: $[X_p, X_{p'}](q) = L_{X_p} X_{p'}(q) = \frac{d}{dt}\Big|_{t=0} \left(\Phi_t^{X_p}\right)^* X_{p'}(q)$

$$= \frac{d}{dt}\Big|_{t=0} \left(d\,\Phi_t^{X_p}\right)_q^{-1} X_{p'}\left(\Phi_t^{X_p}(q)\right)$$

use the chain rule

$$\overset{\vee}{=} \underline{X_{pq - qp}}$$