Exercise Sheet 11

1) We use $L_X Y = \frac{d}{dt}\Big|_{t=0} (\phi_t^X)^* Y \equiv \frac{d}{dt}\Big|_{t=0} (\phi_t)^* Y$

↑ we remove X on the top

$$[X,[Y,Z]] = L_X [Y,Z]$$
$$= \frac{d}{dt}\Big|_{t=0} (\phi_t)^* [Y,Z]$$
$$= \frac{d}{dt}\Big|_{t=0} [(\phi_t)^* Y, (\phi_t)^* Z] = (*)$$

Now:

$$[\phi_t^* Y, \phi_t^* Z](p) = [(d\phi_t(p))^{-1} Y(\phi_t(p)), (d\phi_t(p))^{-1} Z(\phi_t(p))]$$

it j-component is given by: (Einstein convention)

$$[d\phi_t(p)^{-1} Y(\phi_t(p))]^j \cdot \frac{\partial [(d\phi_t(p))^{-1} \cdot Z(\phi_t(p))]^i}{\partial x^j} - [(d\phi_t(p))^{-1} Z(\phi_t(p))]^j \frac{\partial}{\partial x^j} [(d\phi_t(p))^{-1} Y(\phi_t(p))]^i$$

we can denote $[(d\phi_t(p))^{-1} Y(\phi_t(p))]^j$ with $f^j(t,p)$

$[(d\phi_t(p))^{-1} Z(\phi_t(p))]^j$ with $g^j(t,p)$

Then:

$$\frac{d}{dt}\Big|_{t=0} \left(f^j(t,p) \cdot \frac{\partial}{\partial x^j} g^i(t,p) - g^j(t,p) \cdot \frac{\partial}{\partial x^j} f^i(t,p) \right)$$

Leibniz

$$= \underbrace{\left(\frac{d}{dt}\Big|_{t=0} f^j(t,p)\right) \cdot \frac{\partial}{\partial x^j} g^i(0,p)}_{A} + \underbrace{f^j(0,p) \cdot \left(\frac{d}{dt}\Big|_{t=0} \frac{\partial}{\partial x^j} g^i(t,p)\right)}_{B}$$
$$- \underbrace{\left(\frac{d}{dt}\Big|_{t=0} g^j(t,p)\right) \cdot \frac{\partial}{\partial x^j} f^i(0,p)}_{C} - \underbrace{g^j(0,p) \left(\frac{d}{dt}\Big|_{t=0} f^i(t,p)\right)}_{D}$$

$$= (A - D) + (B - C)$$

Since $\phi_0 = id$ we have

$$(*) = \left[\frac{d}{dt}\Big|_{t=0} (\phi_t^*) Y, Z \right] + \left[Y, \frac{d}{dt}\Big|_{t=0} \phi_t^* Z \right]$$

$$= \left[[X, Y], Z \right] + \left[Y, [X, Z] \right]$$

$$= -\left[Z, [X, Y] \right] - \left[[X, Z], Y \right]$$

$\boxtimes$

(2) Rmk: The Jacobi identity may be proven in the following way

Explicit: computation using $[-,-]$

Indirectly: using $L_{(-)}(-) = [-,-]$ (Ex 2)

a) Let $Y \in C^\infty(TN)$, $f: M \to N$ local diffeom, then

$$f^*(Y)(p) := (df_p)^{-1}(Y(f(p))), \quad p \in M.$$

Recall the property of the pullback in Exercise sheet 8, Exercise 4.

Lemma 1: $f: M \to N$ diffeom, then for $X, Y \in C^\infty(N)$,

$$f^*(L_X Y) = L_{f^*(X)} f^*(Y)$$

(or we can simply use $L_X Y = [X,Y]$, $f^*[X,Y] = [f^*X, f^*Y]$.

pf: Set:

$\tilde{X} := f^*(X)$, $\tilde{Y} := f^*(Y)$, $f^*(L_X Y) =: \widetilde{L_X Y}$

Recall that the flow of X is given by a local expression, i.e.

$\forall p \in N$ $\exists$ $\mathcal{U}$ open and $\delta > 0$ s.t.

$$\Phi: \mathcal{U} \times (-\delta, \delta) \longrightarrow N \qquad \text{smooth}$$
$$(x, t)$$

in this notations

$$\begin{cases} \dfrac{\partial \Phi(x,t)}{\partial t} = X(\Phi(x,t)) \\ \Phi(x,0) = x \qquad \text{(fixed } t) \end{cases}$$

we prefer to use the notation $\Phi_t(-)$ for $\Phi(-,t)$. In particular we have

$$p \in N \rightsquigarrow (\mathcal{U}_p, \delta_p, \Phi(t,x): \mathcal{U}_p \times (-\delta_p, \delta_p) \to N)$$
$$\mathcal{U}_p \subset N, \quad \delta_p > 0$$

Now consider $\tilde{\Phi}_t$ defined as follows

. for $q \in M$, set $p := f(q)$, $V_q := f^{-1}(\mathcal{U}_p)$

. let $\tilde{\Phi}$ be the concatenation of:

$$V_q \times (-\delta_p, \delta_p) \xrightarrow{(f \times id)} \mathcal{U}_p \times (-\delta_p, \delta_p) \xrightarrow{\Phi} N \xrightarrow{f^{-1}} M$$

where $(f \times id)(y, t) = (f(y), t)$.

Thus: $\cdot\ \tilde{\Phi} := f^{-1} \circ \Phi \circ (f \times id) : V_q \times (-\delta_p, \delta_p) \longrightarrow M$

$\cdot\ \tilde{\Phi}_t = \tilde{\Phi}(t, -) = f^{-1} \circ \Phi_t \circ f : V_q \longrightarrow M$

WARNING: In the lecture we denote $\Phi(x,t)$ with $\Phi_t(x)$ (even if t is not fixed)

Claim 2: $\tilde{\Phi}$ is the flow of $f^*(X)$.

pf:
$$\frac{\partial}{\partial t} \tilde{\Phi}(x,t) = df^{-1}_{(\Phi \circ (f \times id))} \left(\frac{\partial}{\partial t} (\Phi \circ (f \times id))(x,t) \right)$$
$$= \left(df_{f^{-1} \circ \Phi (f \times id)} \right)^{-1} \circ X \left(\Phi (f \times id)(x,t) \right)$$
$$= \left(df_{\tilde{\Phi}(x,t)} \right)^{-1} \circ X \left(f \left(f^{-1} \left(\Phi (f \times id) \right) (x,t) \right) \right)$$
$$= \left(df_{\tilde{\Phi}(x,t)} \right)^{-1} \circ X \left(f \left(\tilde{\Phi}(x,t) \right) \right)$$
$$= f^*(X) \left(\tilde{\Phi}(t,x) \right) = \tilde{X} \left(\tilde{\Phi}(t,x) \right)$$

Proof of lemma:
$$L_{\tilde{X}} \tilde{Y} = \frac{d}{dt}\Big|_0 \tilde{\Phi}_t^* (\tilde{Y}) \underset{\text{claim}}{=} \frac{d}{dt}\Big|_{t=0} \left(f^{-1} \circ \Phi_t \circ f \right) f^*(Y)$$

property of pull back
$$= \frac{d}{dt}\Big|_{t=0} \left(f \circ f^{-1} \circ \Phi_t \circ f \right) (Y)$$
$$= \frac{d}{dt}\Big|_{t=0} \left(\Phi_t \circ f \right) (Y)$$
$$= \frac{d}{dt}\Big|_{t=0} (f)^* \circ \Phi_t^* (Y)$$
$$= f^* \left(\frac{d}{dt}\Big|_{t=0} \Phi_t^* (Y) \right)$$
$$= f^* (L_X Y)$$

2a) Now let $X, Y \in C^\infty(M)$ s.t. $[X,Y] = 0$

<u>Claim:</u> $(\phi_t^X)^* Y = Y \quad \forall t$ (where defined)

[<u>Mean:</u> let $p \in M$ and let $\phi^X : U_p \times (-\delta_p, \delta_p) \longrightarrow M$ be the flow of X at p, $U_p \subset M$, $p \in U_p$.

$\forall p \in M \quad (\phi_t^X)^* Y = Y \quad , \forall t \in (-\delta_p, \delta_p)$]

<u>pf:</u> To show $\frac{d}{dt}(\phi_t^X)^* Y = Y$ on U_p.

$$\frac{d}{dt}(\phi_t^X)^* Y = \frac{d}{ds}\Big|_{s=t}(\phi_s^X)^* Y = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\phi_{t+\varepsilon}^X)^* Y$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\phi_{\varepsilon+t}^X)^* Y = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\phi_\varepsilon^X)^*(\phi_t^X)^* Y = [X, (\phi_t^X)^* Y]$$

Since $(\phi_0^X)^* X = 0$ and $0 = [X,X] = \frac{d}{dt}\Big|_{t=0}(\phi_t^X)^* X \Rightarrow (\phi_t^X)^* X = X$

<u>Thus:</u> $[X, (\phi_t^X)^* Y] = [(\phi_t^X)^* X, (\phi_t^X)^* Y] = (\phi_t^X)^* \underbrace{[X,Y]}_{0} = 0$

Since $(\phi_0^X)^* Y = Y \Rightarrow (\phi_t^X)^* Y = Y$.

<u>Conclusion:</u> $\phi_t^X \circ \phi_s^Y = \phi_s^Y \circ \phi_t^X$ where are defined

<u>pf:</u> by show $\phi_s^Y = \phi_s^{(\phi_t^X)^* Y}$, since $\phi_t^X : U_p \longrightarrow M$ for a fixed t is a diffeom onto its image, we can use <u>claim 2</u>. $\Phi_s^{(\Phi_t^X)^* Y} = (\Phi_t^X)^{-1} \circ \Phi_s^Y \circ \Phi_t^X$

we conclude

$$\Rightarrow \phi_s^Y = (\phi_t^X)^{-1} \circ \phi_s^Y \circ \phi_t^X$$

Since the proof hold for any $p \in M$, we are done.

2.b) This is theorem 18.6 in lee "intro to C^∞ mfld"

<u>Proposition</u>: $[X,Y]=0$, $p\in M$, $X(p), Y(p)$ lin independent

$\exists\, \mathcal{U}\subset M$, $p\in\mathcal{U}$ and $\psi:\mathcal{U}\to\mathbb{R}^n$ s.t. $\varphi(p)=(y^1(p),\dots,y^n(p))$ with $y^i:\mathcal{U}\to\mathbb{R}$ s.t.

a) ψ is a chart.

b) $X=\frac{\partial}{\partial y^1}$, $Y=\frac{\partial}{\partial y^2}$ in $\mathcal{U}$.

<u>Rmk</u>: This means that for an $f:M\to\mathbb{R}$ $f\in C^\infty(M)$ $q\in\mathcal{U}$

$$X(f)(q)=\underbrace{\frac{\partial}{\partial y^1}f}_{\frac{\partial}{\partial y^1}\left(\underbrace{f\circ\varphi^{-1}}_{\in C^\infty(\varphi(\mathcal{U}),\mathbb{R})}\right)}$$

pf: let $\varphi:\tilde{\mathcal{U}}\to\mathbb{R}^n$ be a chart, $p=\varphi^{-1}(0)$

let $\phi^X:U_p\times(-\delta_1,\delta_1)\to M$, $\phi^Y:V_p\times(-\delta_2,\delta_2)\to M$

<u>Fix</u>: $\tilde{\mathcal{U}}\cap U_p\cap V_p=:\mathcal{U}$, then

$$\phi:(-\delta_1,\delta_1)\times(-\delta_2,\delta_2)\times\mathbb{R}^{n-2}\longrightarrow M$$
$$(x^1,x^2,x^3,\dots,x^n)\longrightarrow\phi^X_{x^1}\circ\phi^Y_{x^2}\circ\varphi^{-1}(0,0,x^3,\dots,x^n)$$

<u>Claim</u>: $d\phi_0:\mathbb{R}^n\longrightarrow T_pM$ is an isom

pf: $\mathbb{R}^n=T_0\mathbb{R}^n$ $\quad \underset{\frac{\partial}{\partial x^1}}{e_1}=(1,0,\dots,0)$, $\underset{\frac{\partial}{\partial x^2}}{e_2}=(0,1,0,\dots,0)$ so on. $\underset{\frac{\partial}{\partial x^j}}{e_j}=(0,\dots,0,\underset{j}{1},0,\dots,0)$

let $\gamma_1:(-\varepsilon,\varepsilon)\to\mathbb{R}^n$, $\gamma_1(t)=(t,0,\dots,0)$, then $\gamma(0)=0$, $\dot\gamma(0)=e_1$

$$d\Phi_0 e_1=\frac{d}{dt}\Big|_{t=0}\Phi(\gamma_1(t))=\frac{d}{dt}\Big|_{t=0}\phi^X_t\circ\underbrace{\phi^Y_0\circ\underbrace{\varphi^{-1}(0,\dots,0)}_{p}}_{=p}=\frac{d}{dt}\Big|_{t=0}\phi^X_t(p)=X(p)$$

let $\gamma_2:(-\varepsilon,\varepsilon)\to\mathbb{R}^n$, $\gamma_2(t)=(0,t,0,\dots,0)$, $\gamma_2(0)=0$, $\dot\gamma_2(0)=e_2$, then

$$d\Phi_0 e_2=\frac{d}{dt}\Big|_{t=0}\Phi(\gamma_2(t))=\frac{d}{dt}\Big|_{t=0}\phi^X_0\circ\phi^Y_t\circ\varphi^{-1}(0,\dots,0)=\frac{d}{dt}\phi^Y_t(p)=Y(p)$$

For $i\geq 2$, let $\gamma_j:(-\varepsilon,\varepsilon)\to\mathbb{R}^n$ $\gamma(t)=(0,\dots,0,\underset{j}{t},0,\dots,0)$: $\gamma(0)=0$, $\dot\gamma(0)=e_j$

$$d\Phi_0(e_j)=\frac{d}{dt}\Big|_{t=0}\Phi(\gamma_j(t))=\frac{d}{dt}\Big|_{t=0}\Phi^X_0\circ\phi^Y_0\circ\varphi^{-1}(0,\dots,0,t,0,\dots,0)=\frac{\partial}{\partial x^j}$$

Note that: $\phi \big|_{(0,0,\mathbb{R}^{n-2})} = \varphi^{-1} \big|_{(0,0,\mathbb{R}^{n-2})}$

since φ is a diffeom we conclude:

1) $\left(\frac{\partial}{\partial x^j}\right)_{2 \leq j \leq n}$ are linearly indep at p

2) $X(p), Y(p)$ lin indep $\Rightarrow$ $X(p), Y(p), \frac{\partial}{\partial x^3}, \dots, \frac{\partial}{\partial x^n}$ are lin indep vectors on T_pM

$\blacksquare$ Claim 1

By inverse funkt thm $\exists\ \tilde{V} \subset \mathcal{U}$ s.t. $p \in \tilde{V}$ and

$$\phi: \tilde{V} \to \phi(\tilde{V})$$

a diffeom. Note that we don't use condition $[X,Y]=0$ for the moment we have proven:

Claim 2: Let X, Y lin independent vector fields, then for any $p \in M$ $\exists\ \psi: \mathcal{U} \to \mathbb{R}^n$ s.t. $\psi(p) = (y_1(p), \dots, y_n(p))$ and

$$X = \frac{\partial}{\partial y^1}, \quad Y = \frac{\partial}{\partial y^2}$$

at p.

(i.e.: $X(p) = \left(\frac{\partial}{\partial y^1}\right)_{p,\psi}$, $Y(p) = \left(\frac{\partial}{\partial y^2}\right)_{p,\psi}$

pf: Let $\psi = \phi^{-1}\big|_{\tilde{V}}$ $\blacksquare$ Claim 2

Now: $0 = [X,Y] \Rightarrow$ we can extend the condition at p to near p

pf of Claim 0: Let $\psi := \phi^{-1}\big|_{\tilde{V}}$, $\psi: \phi(\tilde{V}) \to \tilde{V} \subseteq \mathbb{R}^n$

Let $(y_1, \dots, y_n) \in \tilde{V}$, let $\gamma_1: (-\varepsilon, \varepsilon) \to \tilde{V}$ $\gamma_1(t) = (y_1 + t, y_2, \dots, y_n)$

$q := \phi(y_1, \dots, y_n)$, then

$$\left(\frac{\partial}{\partial y^1}\right)_{q,\psi} = \frac{d}{dt}\Big|_{t=0} \Phi(\gamma_1(t)) = \frac{\partial}{\partial t}\Big|_{t=0} \phi^X_{y_1+t} \circ \phi^Y_{y_2} \circ \varphi^{-1}(0,0,y_3,\dots,y_n)$$

$$= \frac{d}{dt}\Big|_{t=0} \phi^X_t \Big(\underbrace{\circ\, \phi^X_{y_1} \circ \phi^Y_{y_2} \circ \varphi^{-1}(0,0,y_3,\dots,y_n)}_{\varphi^{-1}(y_1, y_2, \dots, y_n) = q} \Big)$$

$$= \frac{d}{dt}\Big|_{t=0} \phi^X_t(q) = X(q)$$

Let $\gamma_i : (-\varepsilon, \varepsilon) \to M$, $\gamma_i(t) = (y_1, y_2, \dots, y_i + t, y_{i+1}, \dots, y_n)$ then

<u>$i=1$</u>

$$\frac{\partial}{\partial y^1}(q) = \frac{d}{dt}\Big|_{t=0} \phi^X_{y_1} \circ \phi^Y_{y_2+t} \circ \varphi^{-1}(0, 0, y_3, \dots, y_n)$$

Crucial fact

$$\overset{2.a)}{=} \frac{d}{dt}\Big|_{t=0} \phi^Y_{y_2+t} \circ \phi^X_{y_1} \circ \varphi^{-1}(0, 0, y_3, \dots, y_n)$$

$$= \frac{d}{dt}\Big|_{t=0} \phi^Y_t \circ \underbrace{\left(\phi^X_{y_1} \circ \phi^Y_{y_2} \circ \varphi^{-1}(0, 0, y_3, \dots, y_n) \right)}_{\varphi^{-1}(y_1, y_2, y_3, \dots, y_n) = q}$$

$$= \frac{d}{dt}\Big|_{t=0} \phi^Y_t(q)$$

$$= Y(q)$$

<u>$i \geq 2$</u>

$$\frac{\partial}{\partial y^i}(q) = \frac{d}{dt}\Big|_{t=0} \phi^X_{y_1} \circ \phi^Y_{y_2} \circ \varphi^{-1}(0, 0, y_3, \dots, y_i + t, \dots, y_n)$$

$$= \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(y_1, y_2, \dots, y_n)$$

USE

3.b) The idea is the <u>same</u>, $\phi : (-\delta_1, \delta_1) \times (-\delta_2, \delta_2) \times \dots \times (-\delta_n, \delta_n) \to M$ given by:

$$\phi(x_1, \dots, x_n) = \phi^{X_1}_{x_1} \circ \phi^{X_2}_{x_2} \circ \dots \circ \phi^{X_n}_{x_n}\left(\varphi^{-1}(0, \dots, 0)\right)$$

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3) Let $X \in C^\infty(TM)$, let $\psi : \mathcal{U} \to \mathbb{R}^n$ be a chart $p \xmapsto{\psi} (x^1, \dots, x^n)$
Then on $\mathcal{U}$ X may be expressed as: $\mathcal{U} \subseteq M \to \mathbb{R}^n$

$$X = \sum X^j(p) \left(\frac{\partial}{\partial x^j}\right)_{p;\psi}$$

where

1) $X^j : \mathcal{U} \to \mathbb{R}$ C^∞

2) $\left(\frac{\partial}{\partial x^j}\right)_p$: directional derivative in direction "x^j" near p

3) $X : C^\infty(M)' \to C^\infty(M)$ via

$$f \mapsto X \circ f$$

$$Xf(p) = X(p).f = \sum X^j(p) \left(\frac{\partial}{\partial x^j}\right)_{p,\psi} f = \sum \underbrace{X^j(\psi^{-1}(x^1, \dots, x^n))}_{C^\infty} . \underbrace{\frac{\partial f \circ \psi^{-1}(x^1, \dots, x^n)}{\partial x^j}}_{C^\infty}$$

s.t. $p = \psi^{-1}(x^1, \dots, x^n)$.

Thus in coordinates we can write $X = \sum X^j(\psi^{-1}(x^1, \dots, x^n)) \frac{\partial}{\partial x^j}$ (little abuse of notation ;))
and in this case $X \in C^\infty(T\psi(\mathcal{U}))$.

A more concrete definition is given

1) Consider the following concatenation of smooth maps

$$\psi(\mathcal{U}) \xrightarrow{\psi^{-1}} \mathcal{U} \xhookrightarrow{i} M$$

2) Then: X in local coordinates is simply $(i \circ \psi^{-1})^* X$ (pull back)

Namely: $(i \circ \psi^{-1})^* X(p) = \left(d(i \circ \psi^{-1})\right)^{-1}_p X(i \circ \psi^{-1})$ $(*)$

Supplementary exercises: Check that the two things agree ☺

Consequence:	M		$\psi(\mathcal{U})$
vector fields:	X	$\rightsquigarrow$	$(i \circ \psi^{-1})^* X$
flows:	ϕ_t^X		$(\psi \circ i^{-1}) \circ \phi_t^X (i \circ \psi^{-1})$

Now by abuse of notation let $X \in C^\infty(T\psi(\mathcal{U}))$ with flow ϕ_t^X (we are in $\mathbb{R}^n$)
By a Taylor series argument we have:

$$\phi_t^X(x_1, \dots, x_n) = \underset{= \phi_0^X(x_1, \dots, x_n)}{(x_1, \dots, x_n)} + \underbrace{t . X(x_1, \dots, x_n)}_{t \cdot \frac{d}{dt}\big|_{t=0} \phi_t(x_1, \dots, x_n)} + \frac{t^2}{2} . \frac{d}{dt^2}\Big|_{t=0} \phi_t^X(x_1, \dots, x_n) +$$

$$+ O(|t|^3)$$ (where $O(p(t))$ is the big O notation) ~~Man~~

Now: consider $\frac{d^2}{dt^2}\Big|_{t=0} \phi_t^X(x_1,\dots,x_n)$, we have

1) $X : M \longrightarrow TM$ smooth map
$p \longrightarrow (p, X(p))$

2) $\phi_t^X : (-\varepsilon,\varepsilon) \longrightarrow M$
$t \longrightarrow \phi_t^X(p)$
for a fixed p may be considered as a path (= integral curve) with $\frac{d}{dt}\Big|_{t=0} \phi_t^X = X(p)$

3) ϕ_t^X is a solution of an ODE s.t.

$$\frac{d}{dt}\phi_t^X = X(\phi_t^X(p)) \implies \frac{d^2}{dt^2}\Big|_{t=0}\phi_t^X = \frac{d}{dt}\Big|_{t=0} X(\phi_t^X(p))$$

Thus we conclude

$$\frac{d^2}{dt^2}\Big|_{t=0}\phi_t^X(p) \underset{③}{=} \frac{d}{dt}\Big|_{t=0} X(\phi_t^X(p)) \underset{\text{"path"}\ ② + ①}{=} (dX)_p\left(\frac{d}{dt}\Big|_{t=0}\phi_t^X(p)\right) \underset{②}{=} (dX)_p\, X(p) = D_X X$$

where $D_X X f = X(Xf)$. Working in $\mathbb{R}^n$ we have:

Thus we have $\phi_t^X(p) = p + tX(p) + \frac{t^2}{2}D_X X(p) + O(t^3)$
where the big O notation means:
$\exists C>0\ \exists t_0>0$ s.t. $\forall t\in[-t_0,t_0]$, $\left|\phi_t^X(p) - p - tX(p) - \frac{t^2}{2}D_X X(p)\right| \leq C\cdot t^3$

Now let consider

$$\phi_s^Y \circ \phi_t^X(p) = \phi_t^X(p) + sY(\phi_t^X(p)) + \frac{s^2}{2} D_Y Y(\phi_t^X(p)) + O(s^3)$$

we expand the first term:

- $\phi_t^X(p) = p + tX + \frac{t^2}{2}D_X X(p) + O(t^3)$

the second:

$$Y(\phi_t^X(p)) = Y(p) + \underbrace{t\,\frac{d}{dt}\Big|_{t=0} Y(\phi_t^X(p))}_{t\cdot D_X X(p)} + O(t^2)$$

(same argument as above)

the last:

$$D_Y Y(\phi_t^X(p)) = D_Y Y p + O(t)$$

(6)

we have:

$$\phi^Y_s \circ \phi^X_t(p) = \phi^X_t(p) + sY(\phi^X_t(p)) + \frac{s^2}{2} D_Y Y(\phi^X_t(p)) + O(s^3)$$

$$= p + tX + \frac{t^2}{2} D_X X(p) + O(t^3)$$

$$+ sY(p) + ts D_X Y(p) + O(st^2)$$

$$+ \frac{s^2}{2} D_Y Y(p) + O(\frac{s^2}{2} \cdot t)$$

[If you want this is just the Taylor formula in 2 variables]

3.a)

$$\Rightarrow \phi^Y_s \circ \phi^X_t(p) = p + tX + \frac{t^2}{2} D_X X(p) + sY(p) + ts D_X Y(p) + \frac{s^2}{2} D_Y Y(p) + O(s^3 + t^3)$$

$$\phi^X_{-t} \circ \phi^Y_s \circ \phi^X_t(p) = \phi_s \circ \phi^X_t(p) - tX(\phi^Y_s \circ \phi^X_t(p)) + \frac{t^2}{2} D_X X(\phi_s \circ \phi^X_t(p)) + O(t^3)$$

again with a Taylor formula.

1 term: $p + tX + \frac{t^2}{2} D_X X(p) + sY(p) + ts D_X Y(p) + \frac{s^2}{2} D_Y Y(p) + O(s^3 + t^3)$

2 term: $X(p) + s \frac{d}{ds}\Big|_{s=0} X(\phi^Y_s \circ \phi^X_0(p)) + t \frac{d}{dt}\Big|_{t=0} X(\phi^Y_0 \circ \phi^X_t(p)) + O(s^2 + t^2)$

(Taylor in two variables)

$$= X(p) + s D_Y X + t D_X X + O(s^2 + t^2)$$

3 term $D_X X(p + O(|t|)) = D_X X(p) + O(|t| + |s|)$

Sum: The above expression is

$$p + tX + \frac{t^2}{2} D_X X(p) + sY(p) + ts D_X Y(p) + \frac{s^2}{2} D_Y Y(p) + O(s^3 + t^3)$$

$$+ -tX(p) - st D_Y X(p) - t^2 D_X X(p) - t\, O(s^2 + t^2)$$

$$+ \frac{t^2}{2} D_X X(p) + \frac{t^2}{2} O(t)$$

$$= p + sY + ts D_X Y + \frac{s^2}{2} D_Y Y + (D_X Y - D_Y X) st + O(|s|^3 + |t|^3)$$

Analogous argument (or if you want Taylor series with 3 variables)
Tells us:

$$\Phi^Y_{-s} \circ \phi^X_{-t} \circ \phi^Y_s \circ \phi^X_t(p) = p + st\left[D_X Y - D_Y X\right] + s^2 D_Y Y - s^2 D_Y Y + O(s^3 + t^3) \quad (3.b)$$

Note:

- The last expression (3.b) is coordinate invariant because so is $\left[D_X Y - D_Y X\right] = [X, Y]$
- The 2nd expression (3.a) is'nt! In particular because $D_X Y$ (with our naive definition is not coordinate invariant)

$\Rightarrow$ to make $D_X Y$ coordinate invariant you need to add other expression Γ called Christoffel symbol

~~Also~~

A more clear way

Let $X, Y \in C^\infty(T\mathbb{R}^n)$, let

$\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ flow of Y

$\phi_s : \mathbb{R}^n \to \mathbb{R}^n$ flow of X

for δ_0 small. look at

$$f : (-\delta_0, \delta_0) \times (-\delta_0, \delta_0) \to \mathbb{R}^n$$
$$(s, t) \longmapsto \phi_s \circ \psi_t(x)$$

Taylor exp of f at $(0;0)$

$$f(s,t) = f(0,0) + s\partial_s f(0,0) + t\partial_t f(0,0) + \frac{s^2}{2}\partial_s^2 f(0,0) + \frac{t^2}{2}\partial_t^2 f(0,0) + ts\,\partial_t\partial_s f(0,0) + O(|t|^3 + |s|^3)$$

Let $p = f(0,0)$, then

$$\partial_t f(0,0) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(0,\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \psi_\varepsilon(p) = Y(p)$$

$$\partial_s f(0,0) = X(p)$$

$$\partial_s^2 f(0,0) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{\frac{d}{d\eta}\Big|_{\eta=\varepsilon} \phi_\eta(p)}_{X(\phi_\varepsilon(p))} = dX_p \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon(p) = dX_p X(p) = D_X X(p)$$

Analogously

$$\partial_s^2 f(0,0) = D_Y Y(p)$$

$$\partial_t \partial_s f(0,0) = \frac{d}{dt}\Big|_{t=0} \underbrace{\frac{d}{ds}\Big|_{s=0} \psi_t \circ \phi_s(p)}_{= X(\phi_s(p))}$$

$$= dX_p \; Y(p) \quad = D_Y X(p)$$

$\Rightarrow$ this prove a)

b) Same idea! Consider the function

$$g(x_1, x_2, x_3, x_4) = \psi_{-x_1} \circ \phi_{-x_2} \circ \psi_{x} \circ \phi_{x_4}(p)$$

Set $f(s,t) = g(t,s,t,s)$

Note $g(t,0,t,0) = p$ $\quad \Rightarrow \quad$ $\partial_s f = \partial_s^2 f = 0$

$g(0,s,0,s) = p$ $\quad\quad\quad\quad$ $\partial_t f = \partial_t^2 f = 0$

By Taylor around 0:

$$f(s,t) = p + st\, \partial_s \partial_t f(0,0) + O(|s|^3 + |t|^3)$$

$$\partial_s \partial_t f(0,0) = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} g(t,s,t,s) \underset{\text{chain rule}}{=} \frac{d}{ds}\Big|_{s=0} (\partial_1 g)(0,s,0,s) + (\partial_3 g)(0,s,0,s) = (*)$$

$$\partial_1 g(0,s,0,s) = \frac{\partial}{\partial x_1}\Big|_{x_1=0} \psi_{-x_1} \circ \underbrace{\phi_{-s} \circ \psi_0 \circ \phi_s(p)}_{=p} = -X(p)$$

$$(\partial_3 g)(0,s,0,s) = d(\psi_{-t} \circ \phi_{-s}) \underbrace{\frac{d}{dx_3} \psi_{x_3} \circ \phi_s(x)}_{= X(\phi_s(p))} = d(\phi_{-s})_{\phi_s(p)} X(\phi_s(p))$$

$$= \phi_s^*(X)(p)$$

Thus $(*) = \frac{d}{ds}\Big|_{s=0} \left[(-X(p) + \phi_s^*(X)(p) \right] = 0 + [X,Y](p)$

4.a) $M = \mathbb{R}^2 \times S^1$

$$X(x_1, x_2, e^{i\theta}) = (\cos\theta, \sin\theta, i \cdot e^{i\theta})$$

$$Y(x_1, x_2, e^{i\theta}) = (\cos\theta, \sin\theta, -i \cdot e^{i\theta})$$

Let $\phi_t^X(x_1, x_2, e^{i\theta})$ be the flow of X with in $\phi_0(x_1, x_2, e^{i\theta}) = (x_1, x_2, e^{i\theta})$

Claim 1 $\phi_t^X(x_1, x_2, e^{i\theta}) = (x_1 - \sin\theta + \sin(t+\theta), x_2 + \cos\theta - \cos(t+\theta), e^{i(t+\theta)})$

pf: $\frac{d}{dt} \phi_t^X(x_1, x_2, e^{i\theta}) = (\cos(t+\theta), \sin(t+\theta), i \cdot e^{i(t+\theta)})$

$$= X(\phi_t^X(x_1, x_2, e^{i\theta}))$$

Analogously:

The flow of Y s.t. $\phi_0^Y(x_1, x_2, e^{i\theta}) = (x_1, x_2, e^{i\theta})$ is

$$\phi_t^Y(x_1, x_2, e^{i\theta}) = (x_1 + \sin\theta - \sin(\theta - t), x_2 - \cos\theta + \cos(\theta - t), e^{i(\theta - t)})$$

By looking at the first two coordinates (in $\mathbb{R}^2$)

Φ_t^X is a rotation around $(x_1 - \sin\theta, x_2 + \cos\theta)$ with $\measuredangle t$ ↺ (counter-clock)

Φ_t^Y " " " $(x_1 + \sin\theta, x_2 - \cos\theta)$ " $\measuredangle t$ ↻ (clock)

b) $XY(p) \underset{\uparrow}{=} dY_p X(p) = \frac{d}{dt}\Big|_{t=0} X(\phi_t^Y(p)) = \frac{d}{dt}\Big|_{t=0} (\cos(\theta - t), \sin(\theta - t), i \cdot e^{i(\theta - t)})$

some explanation as for exercise 3 !

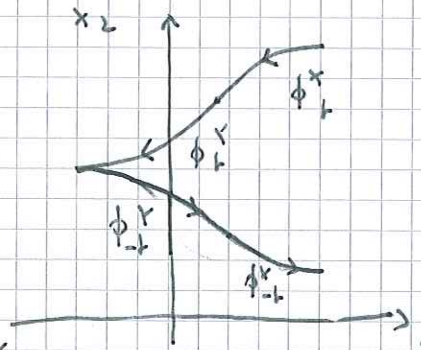
$$= (\sin\theta, -\cos\theta, e^{i\theta})$$

$$YX(p) = dX \cdot Y(p) = \frac{d}{dt}\Big|_{t=0} Y(\phi_t^X(p)) = \frac{d}{dt}\Big|_{t=0} (\cos(\theta + t), \sin(\theta + t), -i e^{i(t+\theta)})$$

$$= (-\sin\theta, \cos\theta, e^{i\theta})$$

$$[X, Y](p) = -dX_p Y(p) + dY_p X(p) = -(2\sin\theta, -2\cos\theta, 0)$$

4.c) $\gamma(t) := \phi^Y_{-t} \circ \phi^X_{-t} \circ \phi^Y_t \circ \phi^X_t$ is a "parking trajectory" (looking only in $\mathbb{R}^2$!)



Explicit

$$(x_1, x_2, e^{i\theta}) \xrightarrow{\phi^X_t} \left(x_1 - \sin\theta + \sin(t+\theta),\ x_2 + \cos\theta - \cos(t+\theta),\ e^{i(t+\theta)}\right)$$

$$\xrightarrow{\phi^Y_t} \Big(x_1 - \sin\theta + \sin(t+\theta) + \sin(t+\theta) - \sin((t+\theta)-t),$$
$$x_2 + \cos\theta - \cos(t+\theta) - \cos(t+\theta) + \cos((t+\theta)-t),$$
$$e^{i((t+\theta)-t)}\Big)$$

$$= \left(x_1 - 2\sin\theta + 2\sin(t+\theta),\ x_2 + 2\cos\theta - 2\cos(t+\theta),\ e^{i\theta}\right)$$

$$\xrightarrow{\phi^X_t} \Big(x_1 - 2\sin\theta + 2\sin(\theta+t) - \sin\theta + \sin(-t+\theta)$$
$$x_2 + 2\cos\theta - 2\cos(\theta+t) + \cos\theta - \cos(-t+\theta)$$
$$e^{i(\theta-t)}\Big)$$

$$\xrightarrow{\phi^Y_{-t}} \Big(x_1 - 3\sin\theta + 2\sin(t+\theta) + \sin(-t+\theta) + \sin(-t+\theta) - \sin(-t+\theta+t)$$
$$x_2 + 3\cos\theta - 2\cos(t+\theta) - \cos(-t+\theta) - \cos(-t+\theta) + \cos(-t+\theta+t)\Big)$$

$$= \left(x_1 - 4\sin\theta(1-\cos(t)),\ x_2 + 4\cos\theta(1-\cos(t)),\ e^{i\theta}\right)$$

By exercise 3.b

$$\gamma(t) = (x_1, x_2, e^{i\theta}) + t^2 \overbrace{(-2\sin\theta, +2\cos\theta, 0)}^{[X,Y]} + O(t^3)$$

Interpretation: think about ☺!