Exercise sheet 12

1) G Lie Group, $X \in C^\infty(TM)$ left invariant VF

a) Let $a(t) := \phi^X_t(e)$, i.e. $\dot{a}(t) = X(a(t))$, $a(0) = e$

Claim: $a(s+t) = a(s)a(t)$

Pf: Fix s, then define $b(t) := a(s)^{-1} a(s+t)$

thus:

$$\dot{b}(t) = \frac{d}{dt} L_{a(s)^{-1}} a(s+t)$$

$$= \left(d L_{a(s)^{-1}}\right)_{a(s+t)} \dot{a}(s+t)$$

$$= \left(d L_{a(s)^{-1}}\right)_{a(s+t)} X(a(s+t))$$

$$= \left(L_{a(s)^{-1}}\right)_* X\left(a(s)^{-1} a(s+t)\right)$$

$$= X(b(t))$$

since $b(0) = a(s)^{-1} a(s) = e \underset{\substack{\uparrow \\ \text{uniqueness of} \\ \text{integral curves}}}{\Longrightarrow} a(t) = b(t) = a(s)^{-1} a(s+t)$.

b) Let X left invariant, let $U \subset G$ be small open set s.t. $\phi^X : (-t,t) \times U \to G$ is def.

Claim: The flow $\phi^X : (-t, +t) \times U \longrightarrow G$ can be extended to $\phi^X : (-t,t) \times G \longrightarrow G$

Pf: if ϕ^X on $(-t,t) \times U$ is defined, then it is also defined on $(-t,t) \times L_{g^{-1}}(U)$ since

$$\phi^X = \phi^{L_g^* X} = \underline{L_g^{-1} \phi_X L_g}$$

Thus we can extend $\phi^X_{\tilde{t}}$ on any $g \in G$, $\tilde{t} \in (-t,t)$

Claim: ϕ^X_t is defined on $\mathbb{R} \times G$

Pf: Suppose that $\exists t \in \mathbb{R}_{>0}$ s.t. $\phi_{\tilde{t}} : (-t,t) \times G \to G$ is defined and t is maximal.

Let $s_1, s_2 < t$, s.t. $s_1 + s_2 > t$ then

$$\phi^X_{t+s}(g) = \phi^X_t\left(\phi^X_s(g)\right) \quad \text{is well def}$$

$\Longrightarrow$ ↯ $\Longrightarrow$ $\underline{t = +\infty}$.

c) **Prop:** $\phi^{sX}_t = \phi^X_{st}$

pf: $\frac{d}{dt}\phi^X_{st}(p) = sX\left(\phi^X_{st}(p)\right)$

since $\phi^X_{st}(p)\Big|_{t=0} = \phi^X_0(p) = p$

$\Longrightarrow$ the two integral curves agree.

d) $\exp : T_e G \longrightarrow G$

$X \longrightarrow a_X(1) = \phi^X_1(e)$

by b), this map is well-defined.

Prop: $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$

pf: Note:

$$\phi^X_t(p) = L_{g^{-1}}\, \phi^X_t(p)\, L_g = g^{-1}\phi^X_t(gp)$$

We have with $g = \phi^X_s(q)$

$$\phi^X_s(q) \cdot \phi^X_t(p) = \phi^X_t\left(\phi^X_s(q)\, p\right) \qquad (*)$$

Thus:
$$\begin{aligned}\exp((s+t)X) &= \phi^{(s+t)X}_1(e)\\ &= \phi^X_{s+t}(e)\\ &= \phi^X_t\left(\phi^X_s(e)\right)\\ &\overset{(*)}{=} \phi^X_s(e)\,\phi^X_t(e)\\ &= \phi^{sX}_1(e)\;\phi^{tX}_1(e)\\ &= \exp(sX)\,\exp(tX)\end{aligned}$$

$\square$

②

Since: $\psi: \mathbb{H} \longrightarrow \mathbb{D}$

$z \longrightarrow \frac{z-1}{z+1}$

is a again a diffeo we have:

$$S^1 \times \mathbb{D} \xrightarrow[\sim]{\psi^{-1}} S^1 \times \mathbb{H} \xrightarrow[\sim]{\phi^{-1}} S^1 \times \mathbb{R}_{\geq 0} \times \mathbb{R} \xrightarrow[\sim]{f} SL(2,\mathbb{R})$$

a chain of diffeomorphism.

d) $\exp: sl(2,\mathbb{R}) \longrightarrow SL(2,\mathbb{R})$ is not surj.

Pf: Consider $B \in SL(2,\mathbb{R})$

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$\det(B - \lambda \mathbb{1}) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - tr(B)\lambda + 1$

$\Rightarrow \lambda_1, \lambda_2 = \frac{tr(B) \pm \sqrt{tr(B)^2 - 4}}{2}$

Thus: $\cdot$ if $|tr(B)| < 2 \Rightarrow |\lambda_1| = |\lambda_2| = 1 \Rightarrow B \sim$ rotation

$\cdot$ if $|tr(B)| > 2$: since $\sqrt{tr(B)^2 - 4} < |tr(B)|$

$\Rightarrow$ if $\lambda_1 > 0 \Rightarrow \lambda_2 > 0$, if $\lambda_1 < 0 \Rightarrow \lambda_2 < 0$

Now: suppose that B satisfies $\lambda_1 < 0, \lambda_2 < 0$, suppose that $\exists A \in sl(2,\mathbb{R})$ s.t. $\exp(A) = B$

now since $tr(A) = 0$ by Cayley Hamilton then

$$A^2 = -\det(A) \cdot I$$

i) if $\det(A) = 0 \Rightarrow A^2 = 0 \Rightarrow A \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\exp(A)$ is conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

i.e. a matrix with only 1 eigenvalue 1.

ii) if $\det(A) > 0$, then the eigenvalues of A are λ and $-\lambda$ ($trA = 0$!) s.t. $-\lambda^2 > 0 \Rightarrow \lambda$ is purely imaginary with $\lambda = it$

2) a) Let A be a complex (or real) square matrix, then:

$$\det(\exp(A)) = \exp(\mathrm{tr}(A))$$

pf: this is just a calculation. The solution is:
- at wikipedia entry (Jacobi's formula for matrices) ☺

It can be solved via an explicit calculation

b) This is totally similar to other matrices-Lie groups (see previous exercises)

c) Recall that any $g \in SL(2,\mathbb{R})$ can be written uniquely as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = (*)$$

$$= \begin{pmatrix} r\cos\theta & xr\cos\theta - (\sin\theta)/r \\ r\sin\theta & xr\sin\theta - (\cos\theta)/r \end{pmatrix}$$

where: $r = \sqrt{a^2+b^2} > 0$, $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{c}{r}$

and $x = \dfrac{ab+cd}{a^2+c^2}$

By computation we have:
① such a decomposition is well defined
② it is unique (check!)

Now consider:

$$f: S^1 \times \mathbb{R}_{>0} \times \mathbb{R} \longrightarrow SL(2,\mathbb{R})$$
$$(\theta, r, x) \longmapsto (*)$$

this map is clearly smooth + inj + surj.

$\Rightarrow f$ is a diffeo

Now we have a diffeo $\phi: \mathbb{R}_{\geq 0} \times \mathbb{R} \simeq \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$

$$(a, b) \longmapsto b + i \cdot a$$

thus $\exp(A)$ has eigenvalues $e^{i\lambda}$, $e^{-i\lambda}$

iii) if $\det A < 0$ then A has two eigenvalues $\lambda, -\lambda$ with $-\lambda^2 < 0$
then the eigenvalue of $\exp(A)$ are $e^{\lambda}, e^{-\lambda} > 0$.

$\Longrightarrow$ thus we have 3 contradictions!

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