Since $\text{dist}_{S^{n-1}}\left(\tau(s_0), \tau(t)\right)$ = angle between $\tau(s_0)$, $\tau(t)$ in the plane spanned by $\tau(s_0)$, $\tau(t)$

$\Rightarrow \cos\left(\text{dist}_{S^{n-1}}\left(\tau(s_0), \tau(t)\right)\right) = \tau(s_0) \cdot \tau(t)$

$\Rightarrow$ (*) is equivalent to

$$< \tau(t), \tau(s_0) > \; > 0 \quad \forall\, t \in [0, L]$$

Wh WLOG $\tau(s_0) = (0, 0, \dots, 1)$ and define

$$h(s) := < \gamma(s), (0, 0, \dots, 1) >$$

Since $\gamma(0) = \gamma(L)$, we have

$$0 = h(L) - h(0) = \int_0^L \frac{dh}{ds}\, ds = \int_0^L \underbrace{< \frac{d}{ds}\gamma, (0, \dots, 1)}_{>0}\, ds > 0$$

$\Rightarrow$ ↯ $\boxtimes$ Claim

Hence: $\frac{\pi}{2} \leq \text{dist}_{S^{n-1}}\left(\tau(s), \tau(t)\right)$

$$\leq \int_s^t \left|\frac{d\tau}{ds}\right| ds$$

shortest path argument

$$= \int_s^t |k(s)|\, ds \leq \int_0^L |k(s)|\, ds = \int_\gamma |k(s)|\, ds$$

$\boxtimes$