1b) **Prop:** Let $\gamma : [0,L] \to \mathbb{R}^2$ be the a closed curve in $\mathbb{R}^2$, then

$$\int_\gamma |K| \geq 2\pi$$

"pf": The general proof is really difficult!

Here see special case:

Idea: ① Since $\tau : [0,L] \to S^1$, then exist a function

$$\theta : [0,L] \to \mathbb{R} \quad \text{s.t.}$$

$$\tau(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}$$

Since: • $\gamma(0) = \gamma(L) \implies \theta(0) - \theta(L) \in 2\pi\mathbb{Z}$

" • $\dfrac{d\tau(s)}{ds} = \dfrac{d\theta(s)}{ds} N(s)$ ← normal vector of $\gamma(s)$

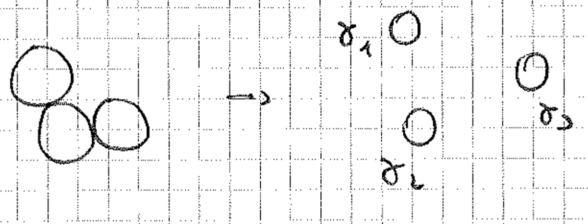
$$\implies K(s) = \frac{d\theta(s)}{ds}$$

$$\implies \int_\gamma K = \int_0^L \frac{\partial}{\partial s}\theta(s) = \theta(0) - \theta(L) \in 2\pi\mathbb{Z}$$

Consequence: • if γ is a closed curve, hence s.t. $\int_\gamma K \neq 0 \implies$ done!

• if γ can be decompose in **smooth** curves, then

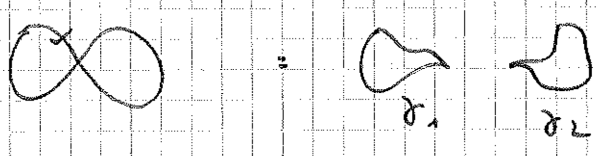
Ex: γ :



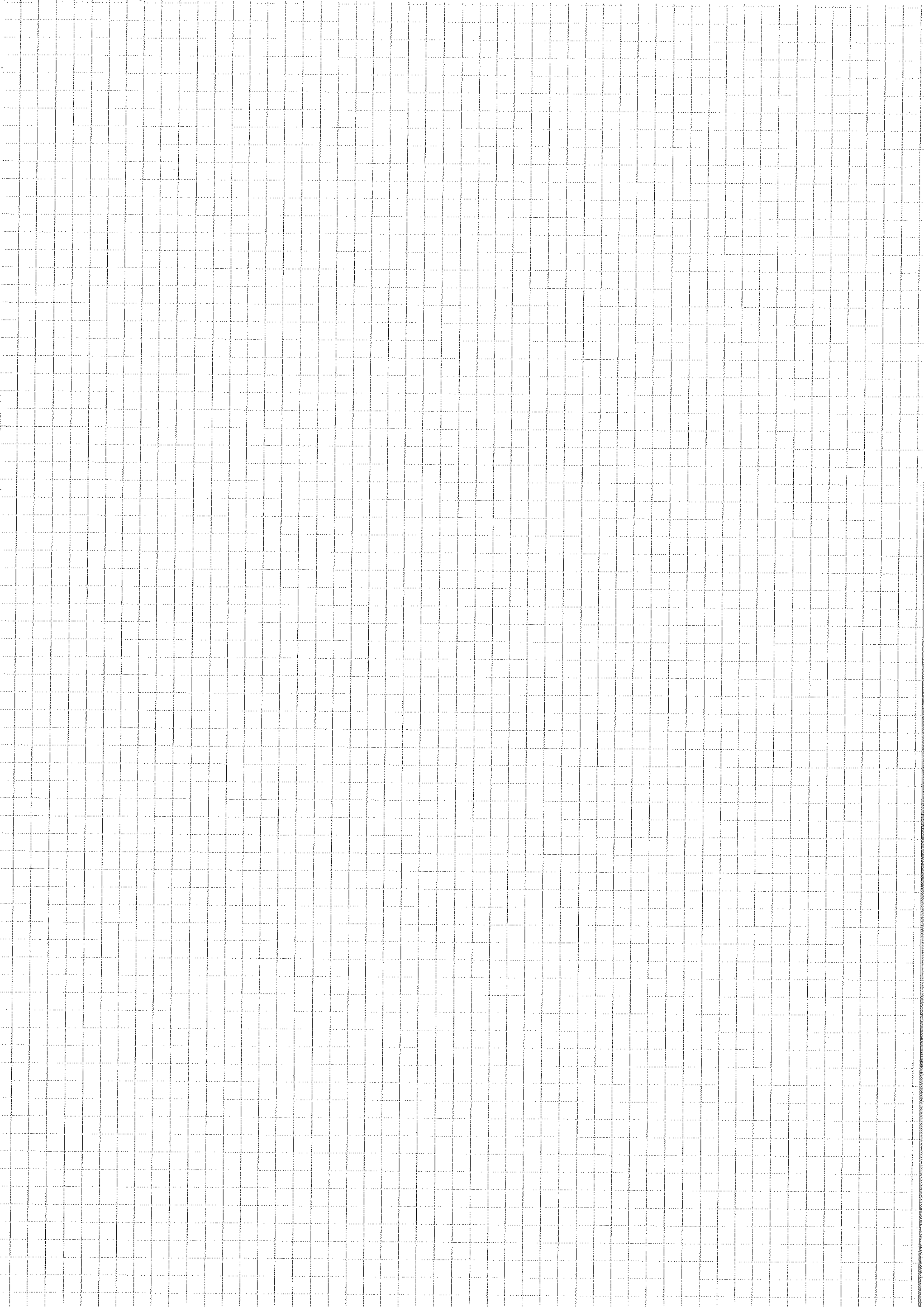
$$\implies \int_\gamma |K| = \int_{\gamma_1} |K| + \int_{\gamma_2} |K| + \int_{\gamma_3} |K| = 2\pi|n|$$

Hard case: γ decompose into piecewise smooth curves

Ex: γ :



I don't know a **solution**! ☹



d) Note that:

$$\dot{\tau} = \frac{d\tau}{ds} = K = kN$$

$$\dot{N} = \langle \dot{N}, \tau \rangle \tau + \langle \dot{N}, B \rangle B$$

$$= \langle \frac{d}{ds} \frac{K}{|K|}, \tau \rangle \tau + \ell B$$

$$= \frac{1}{|K|} \langle \dot{K}, \tau \rangle \tau + \ell B$$

$$= -\frac{1}{|K|} \langle K, \dot{\tau} \rangle \tau + \ell B = \ell B - k\tau$$

$$\dot{B} = \frac{d}{ds} (\tau \times N) = \dot{\tau} \times N + \tau \times \dot{N}$$

$$= kN \times N + \tau \times (\ell B - k\tau)$$

$$= \ell(-N) + (-k)(\tau \times \tau)$$

$$= -\ell N$$

Thus we have

$$\frac{d}{ds} \begin{pmatrix} \tau \\ N \\ B \end{pmatrix} = \begin{pmatrix} kN \\ \ell B - k\tau \\ -\ell N \end{pmatrix} \quad (*)$$

$$= \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \ell \\ 0 & -\ell & 0 \end{pmatrix} \begin{pmatrix} \tau \\ N \\ B \end{pmatrix} \qquad \text{define } M(s) := \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \ell(s) \\ 0 & -\ell(s) & 0 \end{pmatrix}$$

<u>Note:</u> $\tau(s), B(s), N(s)$ define a ONB for $\mathbb{R}^3$ for any s

Suppose that $\tau(s), B(s), N(s)$ are defined for $s \in [a,b]$ and solved $(*)$, then let

$$S(s) = \big(\tau(s), N(s), B(s)\big) \quad \text{be a function}$$

$$S : [a,b] \longrightarrow M^{3\times 3}(\mathbb{R})$$

<u>Then:</u> $\cdot \; \frac{d}{ds} S(s) = S(s)\, M(s)$

$\cdot \; \frac{d}{ds} S(s)^* = \big(M(s) S(s)\big)^* = -M(s)\, S(s)^* \qquad (M(s) \text{ is skew symm.})$

$$\Longrightarrow \frac{d}{ds}\left(S S^*\right) = S \frac{dS^*}{ds} + \frac{dS}{ds} S^* = -SMS^* + SMS^* = 0$$

Exercise 2

a) Calculation:

$\kappa(s) = 2a, \quad \ell(s) = \frac{3b}{a}$

b) we use 2d), since $\ell = 0$, then the DG (*) is:

$$\frac{d}{ds}\begin{pmatrix} T \\ N \end{pmatrix} = \begin{pmatrix} 0 & k \\ -1/k & 0 \end{pmatrix}\begin{pmatrix} T \\ N \end{pmatrix}, \quad \frac{d}{ds} B = 0$$

$\Rightarrow \quad T, N \in \operatorname{span}\{T(0), N(0)\}$

c) we use 2d)

① Let $K > 0$, $\ell \in \mathbb{R}$

$\gamma_E(t) = (r\cos(t), r\sin(t), bt)$ be the helix.

since $\dot{\gamma}_E(t) = (-r\sin(t), r\cos(t), b)$, $|\dot{\gamma}_E(t)| = \sqrt{r^2+b^2}$

$$\Rightarrow \quad T = \frac{(-r\sin(t), r\cos(t), b)}{\sqrt{r^2+b^2}}$$

$$\mathcal{K} = \frac{(-r\cos(t), -r\sin(t), 0)}{r^2+b^2}, \quad N = (-\cos(t), \sin(t))$$

$K = \frac{r}{r^2+b^2} > 0 \Leftrightarrow r > 0$, By calculation

$$B = \frac{1}{\sqrt{r^2+b^2}}\begin{pmatrix} b\sin(t) \\ -b\cos(t) \\ r \end{pmatrix}, \quad \ell = \frac{b}{r^2+b^2}$$

$$K^2+\ell^2 = \frac{1}{r^2+b^2} \Rightarrow r = \frac{K}{K^2+\ell^2}, \quad b = \frac{\ell}{K^2+\ell^2}$$

Thus: $\gamma_E \Rightarrow$ $K = \text{const}$, $\ell = \text{const}$.

② Now: Put γ_E in the DG of 2.d) with $\ell = \text{const}$, $K = \text{const}$.

$\Rightarrow$ conclude that γ_E is a solution of such a DG and use the uniqueness.

Thus $g g^*(s)$ is constant.
Since (using the initial condition) there is a $s \in [a,b]$ s.t. $g(s) g(s)^* = Id$
$\Rightarrow g g^*(s) = id \quad \forall \quad s \in [a,b]$.

Consequence:
- The solution of (*) are preserved via el of $O(3)$
- Since they are clearly preserve by translations
  $\Rightarrow$ they are preserved by a <u>rigid motion</u> !

<u>Exercise 3</u> : simple topological practice

<u>Exercise 4</u> : solved in <u>exercise sheet 2</u> (look at the Hint!)