Ex Sheet 9

① **Show:** $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Y.f)X$ ①

pf: Let $X \in TM$, then for $p \in M$ $\psi : \mathcal{U} \to \mathbb{R}^n$ chart we can write

$$X(p) = \sum X^j(p)\left(\frac{\partial}{\partial x^j}\right)_{p,\psi}$$

Thus <u>locally</u> a vector field may be viewed as

$$X = \sum \underbrace{X^i}_{C^\infty \text{ fnct on } \mathcal{U} \subset M} \underbrace{\frac{\partial}{\partial x^i}}_{\text{element of } T\mathcal{U}}$$

$h, f, g \in C^\infty(M)$, $X, Y \in TM$ then

$$X_g Y h = \sum_i X^i \frac{\partial}{\partial x^i}\left(g \cdot \sum_j Y^j \frac{\partial}{\partial y^j} h\right)$$

Leibniz Rule

$$= \sum_i X^i\left(\frac{\partial}{\partial x^i} g\right)\cdot\left[\sum_j Y^j \frac{\partial}{\partial y^j} h\right] + \sum_i X^i g \left(\sum_j \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial}{\partial y^j} \cdot h\right)\right)$$

$$= \sum_i X^i \left(\frac{\partial}{\partial x^i} g\right)\left(\sum_j Y^j \frac{\partial}{\partial y^j} h\right) + \sum_i X^i g \left(\sum_j \frac{\partial}{\partial x^i}(Y^j) \frac{\partial}{\partial y^j} h\right)$$

$$+ \sum_i X^i g \cdot \left(\sum_j Y^j \cdot \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial y^j} h\right)\right)$$

We have a similar expression for $Y f X h$, Moreover we have

$$X g \cdot Y h = (X(g)) \cdot (Y(h)) + g(X(Y(h)))$$

we conclude:

$$[fX, gY]h = f\cdot X(g(Y(h))) - g\cdot Y(f(X(h)))$$

$$= f(Xg)\cdot(Yh) + fg\cdot(X(Y(h))) - g(X(h))(Y(f)) - fg(YX(h))$$

$$= fg[X,Y](h) + f(X(g))Y(h) - g(Y.f)\cdot(X)(h)$$

2) a) A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called locally Lipschitz continuous if $\forall p \in A$ there exists $\mathcal{U} \subset A$ open, $L_{\mathcal{U}} > 0$ s.t.

$$\| f(x) - f(y) \| \leq L_{\mathcal{U}} \| x - y \| \quad \forall x, y \in \mathcal{U}$$

Consider the system

$$* \begin{cases} \frac{d}{dt} \gamma(t) = X(\gamma(t)) \\ \gamma(0) = y_0 \end{cases} \qquad X : \mathbb{R}^n \to \mathbb{R}^n \text{ locally Lipschitz continuous}$$

$t \in [0, T)$

__Claim:__ let $\gamma, \tilde{\gamma}$ be two solutions of $(*)$ in $C^1([0,T), \mathbb{R}^n)$. Then $\gamma = \tilde{\gamma}$.

__pf:__ Consider the function $h : [0, t) \to \mathbb{R}^n \times \mathbb{R}^n$, $t \mapsto (\gamma(t), \tilde{\gamma}(t))$

it is C^1 (hence C^0). Since the set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x = y\}$ is closed so is

$$A := \{(x,y) \mid x = y\} \cap h([0,T))$$

in particular $I := h^{-1}(A) = \{ t \in [0,t) \mid \gamma(t) = \tilde{\gamma}(t) \}$ is closed, $0 \in I$. Now let $t_0 \in I$, let $p = \gamma(t_0) = \tilde{\gamma}(t_0)$, let $\mathcal{U}$ be an open set in $\mathbb{R}^n$ of p s.t.

$$\| X(x) - X(y) \| \leq L_{\mathcal{U}} \| x - y \| \quad \forall x, y \in \mathcal{U}.$$

Since $\gamma, \tilde{\gamma}$ are C^1 (hence C^0), $\mathcal{U}$ open there exist a $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$$\gamma((t_0 - \delta, t_0 + \delta) \cap [0,T)) \subset \mathcal{U}$$
$$\tilde{\gamma}((t_1 - \delta, t_1 + \delta) \cap [0,T)) \subset \mathcal{U}$$

set $\delta = \min(\delta_1, \delta_2)$, then for $t \in (t_0 - \delta, t_0 + \delta) \cap [0,T)$

$$\frac{d}{dt} \| \gamma(t) - \tilde{\gamma}(t) \|^2 = 2 \langle \gamma(t) - \tilde{\gamma}(t), \dot{\gamma}(t) - \dot{\tilde{\gamma}}(t) \rangle$$

$$\underset{\text{CAUCHY-SCHWARZ}}{\leq} 2 \| \gamma(t) - \tilde{\gamma}(t) \| \| \dot{\gamma}(t) - \dot{\tilde{\gamma}}(t) \|$$

$$= 2 \| \gamma(t) - \tilde{\gamma}(t) \| \cdot \| X(\gamma(t)) - X(\tilde{\gamma}(t)) \| \quad \text{since } \dot{\tilde{\gamma}}(t) = X(\tilde{\gamma}(t)), \ \dot{\gamma}(t) = X(\gamma(t))$$

$$\underset{\text{Lipschitz}}{\leq} 2 L_{\mathcal{U}} \| \gamma(t) - \tilde{\gamma}(t) \|^2$$

__Now__ $f(t) := e^{-2 L_{\mathcal{U}} t} \| \gamma(t) - \tilde{\gamma}(t) \|^2 \geq 0$, $f : (t_0 - \delta, t_0 + \delta) \cap [0,T) \to \mathbb{R}$

Then:

- $f(t_0) = e^{-2Lu t_0} \underbrace{\|\gamma(t_0) - \tilde{\gamma}(t_0)\|}_{=0} = 0$

- $df(t) = -2Lu\, f(t) + e^{-2Lu t_0} \frac{d}{dt}\left(\|\gamma(t) - \tilde{\gamma}(t)\|^2\right)$

  $\leq -2Lu\, f(t) + 2Lu\, f(t) = 0$

$\Rightarrow f : (t_0-\delta, t_0+\delta) \cap [0,T) \to \mathbb{R}$ constant,

i.e. for any $t \in I$ there exist a $\delta>0$ s.t. $[0,1) \cap (t_0-\delta, t_0+\delta) \subset I \Rightarrow I$ open

$\Rightarrow I$ is open and closed in $[0,T)$, $[0,T)$ connected $\Rightarrow I = [0,T)$

~~Rmk: The same proof may be used for $(-T,0]$ to conclude $f(t) = e$~~

3) Rmk: It is sufficient to work with $M = \mathbb{R}^n$. This because differentiability and continuity are local properties, thus if we do a chart $\varphi: U \to \mathbb{R}^n$ we have

- $\sum X^i \frac{\partial}{\partial x^i} \in C^k(TU) \iff X^1, \dots, X^n \in C^k(U, \mathbb{R}^1) \iff X^1 \circ \varphi^{-1}, \dots, X^n \circ \varphi \in C^k(\mathbb{R}^n, \mathbb{R}^n)$

- $\gamma \in C^k(I, M) \iff \varphi \circ \gamma \in C^k(I, \mathbb{R}^n)$

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- $\sum X^i \frac{\partial}{\partial x^i} \in C^k(TU) \iff X^1, \dots, X^n \in C^k(U, \mathbb{R}^n)$

$\iff X^1 \circ \varphi^{-1}, \dots, X^n \circ \varphi^{-1} \in C^k(\mathbb{R}^n, \mathbb{R}^n)$

- $\gamma \in C^k(I, U) \iff \varphi \circ \gamma \in C^k(I, \mathbb{R}^n)$

If $\gamma \in C^1(I, \mathbb{R}^n)$, $X \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, then

$$\dot{\gamma} = \underbrace{X \circ \gamma}_{\in C^1(\mathbb{R}, \mathbb{R}^n)}$$

$$\Rightarrow \gamma \in C^2(I, \mathbb{R}^n) \quad \gamma(t) = \gamma(0) + \int_0^t \dot{\gamma}(t)\, dt$$

$$= \gamma(0) + \int_0^t X(\gamma)\, dt$$

if $k \geq 2$ and the $\dot{\gamma} = \underbrace{X \circ \gamma}_{\in C^2(I, \mathbb{R}^n)}$

$$\Rightarrow \gamma \in C^3(\mathbb{R}, \mathbb{R}^n), \dots$$

4. a)

Assuming f smooth and bijective.
Then for any $q \in N$ we have that

$$\underbrace{df_{f^{-1}(q)}}_{\text{map from } T_{f^{-1}(q)}M \to T_q N} \overbrace{X(f^{-1}(q))}^{\in T_{f^{-1}(q)}M}$$

is an element of T_qN, hence we have a map

$$f_*(X) : N \longrightarrow T_qN$$

Suppose that f is a diffeomorphism since

- $df : TM \to TN$ is smooth
  $(p, X_p) \longmapsto (f(p), df_p X_p)$
- $X : M \to TM$ is smooth

The composition

$$N \xrightarrow{f^{-1}} M \xrightarrow{X} TM \xrightarrow{df} TN$$

is again smooth

b) similar to a) the only difference is that (df_p^{-1}) exists always, hence df_p is an isom. for any $p \in M$ and thus f is a local diffeom.

c)
$$\begin{aligned}
(g\circ f)^* X(p) &= d(g\circ f)_p^{-1} X(g\circ f(p)) \\
&= (dg_{f(p)} \circ df_p)^{-1} X(g\circ f(p)) \\
&= df_p^{-1} \circ dg_{f(p)}^{-1} (X(g(f(p)))) \\
&= df_p^{-1}\, g^* X(f(p)) \\
&= f^*(g^*(X))(p)
\end{aligned}$$

here: $K \xrightarrow{f} N \xrightarrow{g} M$, $X \in C^\infty(M)$; $p \in K$

Let $\underset{\substack{\in\\ q}}{K} \longrightarrow N \longrightarrow M$ maps and $X \in C^\infty(K)$, then

$$(g\circ f)_* X(q) = d(g\circ f)_{(g\circ f)^{-1}(q)} X\left((g\circ f)^{-1}(q)\right)$$

$$= dg_{g^{-1}(p)} \circ df_{(g\circ f)^{-1}(q)} X\left(f^{-1}(g^{-1}(q))\right)$$

$$= dg_{g^{-1}(p)} f_*(X)\left(g^{-1}(q)\right)$$

$$= g_* f_*(q)$$

Now we have by chain rule

$$df_{f^{-1}(q)} = \left(df^{-1}_q\right)^{-1}, \quad df^{-1}_{f(p)} = \left(df_p\right)^{-1}$$

Then $f_*(X)(q) = df_{f^{-1}(q)} X(f^{-1}(q)) = \left(df^{-1}_q\right)^{-1} X(f(p)) = \left(f^{-1}\right)^* X(q)$

$$\left(f^{-1}\right)_*(X)(p) = d\left(f^{-1}\right)_{f(p)} X(f(p)) = \left(df_p\right)^{-1} X(f(p)) = f^*(X(p))$$

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a) Consider the vector field

$$T_v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longrightarrow v$$

it is constant and $f \in C^\infty(\mathbb{R}^3)$ $T_v(f) =$ "directional derivative of f along v"

Then $T_v T_w (f) = T_w T_v (f)$

$\Rightarrow [T_v, T_w] = 0$

Let $R_v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$x \longrightarrow V \times X = \begin{pmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{pmatrix}$$

Recall that $X, Y \in C^\infty(TM)$,

$$[X,Y]^i = \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

Then:

$$[T_v, R_w]^1 = v_1 (0) + -v_2 \cdot w_3 + v_3 \cdot w_2$$
$$[T_v, R_w]^2 = -v_3 w_1 + v_1 w_3 \quad = -(w \times v) = v \times w$$
$$[T_v, R_w]^3 = -v_1 w_2 + v_2 w_1 \quad = T_{v \times w}$$

Analogously we have

$$[R_v, R_w] = R_{v \times w}$$

b) By above we have

$[T_i, T_j] = 0 \quad \forall i,j$

$$[R_1, R_2](x) = \left(\frac{\partial}{\partial x^1} \times \frac{\partial}{\partial x^2} \right) \times x = \frac{\partial}{\partial x^3} \times x = R_3(x)$$

$$[R_3, R_1](x) = \left(\frac{\partial}{\partial x^3} \times \frac{\partial}{\partial x^1} \right) \times x = \frac{\partial}{\partial x^2} \times x = R_2(x)$$

Since $[R_i, R_j] = -[R_j, R_i]$ we have

$$[R_i, R_j] = a_{ij} R_k$$

s.t. $a_{ij} = -a_{ji}$

- $a_{12} = a_{31} = 1$
- $a_{23} = 1$

, $k \in \{1,2,3\}$, $k \neq i$, $k \neq j$

3) remains to prove

$$[T_i, R_j](x) = \left(\frac{\partial}{\partial x_i} \times \frac{\partial}{\partial x_j}\right)$$

Thm:

$$[T_i, R_j] = b_{ij} T_k \quad , \quad \begin{array}{l} \cdot\ b_{ij} = -b_{ji} \\ \cdot\ b_{12} = b_{31} = 1 \\ \cdot\ b_{23} = 1 \end{array} \qquad \begin{array}{l} k \in \{1,2,3\} \\ \text{s.t. } k \neq i,\ k \neq j. \end{array}$$