Differential Geometry

D-MATH, HS 2014

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Supplementary Exercise

1. (a) Define for $\alpha := (j, \sigma)$ where $j = 1, \ldots, n+1$, $\sigma \in \{+, -\}$ the sets

$$U^{j,+} := S^n \cap \left\{x^j > 0\right\}, \quad U^{j,-} := S^n \cap \left\{x^j < 0\right\}.$$

Consider the maps

$$\begin{array}{rcl} \phi^{j,\pm} : U^{j,\pm} & \to & \mathbb{R}^n \\ \left(x^1, \ldots, x^{n+1}\right) & \mapsto & \left(x^1, \ldots, x^{j-1}, x^j \ldots, x^{n+1}\right) \end{array}$$

Let $\mathcal{A}_1 := \{(\phi^\alpha, U^\alpha)\}$. Show that it is an atlas on S^n.

(b) Let $N^+ := (0, 0, \ldots, 0, 1)$, $N^- := (0, 0, \ldots, 0, -1)$. Note that the two stereograpic projections

$$\psi^+ : V^+ := S^n / \left\{N^+\right\} \to \mathbb{R}^n, \quad \psi^- : V^- := S^n / \left\{N^-\right\} \to \mathbb{R}^n$$

defined by

$$\psi^\pm \left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^{n+1}\right)}{1 \pm x^{n+1}},$$

are bijections. Show that $\mathcal{A}_2 := \{(\psi^\pm, V^\pm)\}$ is an atlas on S^n.

(c) Show that the two atlases are equivalent.

2. Let $\mathbb{RP}^n := \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$. For $p \neq 0$ in $\mathbb{R}^{n+1}$, let $[p]$ be the line through p and 0. Show that the map $\pi : S^n \to \mathbb{RP}^n$, $\pi(p) := [p]$ is smooth.

3. Let M be a set, $\mathcal{A}$ be an atlas on M, and $\bar{\mathcal{A}}$ the associated maximal atlas. Show $\mathcal{A}$ and $\bar{\mathcal{A}}$ induce the same topology on M, i.e. $\mathcal{J}_{\mathcal{A}} = \mathcal{J}_{\bar{\mathcal{A}}}$.

4. Let $(M, \mathcal{A}_M)$, $(N, \mathcal{A}_N)$ be smooth manifolds. Recall the definition in class of an atlas $\mathcal{A}_{M \times N}$ for the cartesian product $M \times N$ by the specification

$$\mathcal{A}_{M \times N} := \{(U \times V, (\phi, \psi)) | (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}.$$

(a) Verify $\mathcal{A}_{M \times N}$ is an atlas and $(M \times N, \bar{\mathcal{A}}_{M \times N})$ is a smooth manifold. Is $\mathcal{A}_{M \times N}$ maximal?

(b) Prove the canonical projection maps

$$\pi_M : M \times N \to M$$
$$\pi_N : M \times N \to N$$

are smooth.

5. (a) Let M_1 be the configuration space of all triangles in the plane with side lengths 3, 4 and 5. What manifold is this?

(b) Let M_2 be the configuration space of all equilateral triangles in the plane with side length 1. What manifold is this?

6. The Möbius band M is the strip $S := (0,3) \times (0,1)$ identified with itself via the equivalence relation characterized by

$$(x, y) \sim (x+2, 1-y)$$

whenever the two points lie in S, that is, $M := S/\sim$. Give M the structure of a smooth manifold by specifying an atlas consisting of two charts.

7. Let $(M^n, \mathcal{A})$, $(N^m, \mathcal{B})$ be smooth manifolds, $f : M \to N$. Show: f is smooth at x in some chart iff it is smooth in all charts, i.e. TFAE

(a) there exists charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ and a open set $W \subseteq U$ such that $x \in W, f(W) \subseteq V$, and

$$\psi \circ f \circ \phi^{-1}|_{\phi(W)} : \phi(W) \subseteq \mathbb{R}^n \to \mathbb{R}^m,$$

is smooth.

(b) For all charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ with $x \in U, f(x) \in V$, there exists an open set $W \subseteq U$ such that $x \in W, f(W) \subseteq V$, and

$$\psi \circ f \circ \phi^{-1}|_{\phi(W)} : \phi(W) \subseteq \mathbb{R}^n \to \mathbb{R}^m,$$

is smooth.

8. Let M be a smooth manifold and $\psi : U \to \mathbb{R}^n$ a chart for M. Let

$$\left(\frac{\partial}{\partial x^i}\right)_{p,\psi} \in T_pM, \quad p \in U, \quad i = 1, \ldots, n$$

be the coordinate vector fields on U induced by the chart ψ. Prove that T_pM is a vector space with basis $\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$ by establishing

(a) prove that

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$$

are linearly independent.

(b) Prove that any $X \in T_pM$ can be expressed as a linear combination of

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$$

(c) Prove that any linear combination of

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$$

lies in T_pM.

This completes the proof that T_pM is a vector space with basis

$$\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p.$$

9. Let M be a smooth manifold with atlas $\mathcal{A}_M = \{(U, \phi)\}$.

(a) Construct a corresponding atlas $\mathcal{A}_{TM} = \{(\mathcal{U}, \Phi)\}$ for the tangent bundle TM of M (repeat the definition from class).

(b) Prove $\mathcal{A}_{TM}$ is an atlas and $(TM, \mathcal{A}_{TM})$ is a smooth manifold.

10. Let M be a smooth manifold, and let D be the set $\cup_{p \in M} D_p$, where D_p is the set of orientations of T_pM (a two-element set).

(a) Show that D naturally has the structure of a smooth manifold with a covering map $D \to M$ of degree 2. D is called the *orientation double cover* of M.

(b) Show that an orientation of M corresponds to a continuous section of the covering map $D \to M$ (that is, a map $f : M \to D$ such that $f(p) \in D_p$ for each p).

(c) In particular, M is orientable if and only if D is diffeomorphic to the product $M \times \{0, 1\}$ (as covering space of M).

(d) Show that D is oriented in a natural way.

11. Let $(M, \mathcal{A}_M)$ be a smooth manifold. Let $(N, \mathcal{A}_N)$ be a submanifold of M. Let $\mathcal{J}_N$ be the topology on M induced by $\mathcal{A}_M$). Prove: the topology induced on N by $\mathcal{A}_N$ (the atlas topology) coincide with the topology induced on N by $\mathcal{J}_M$ (the subspace topology).

12. (Continued form exercise sheet 3, exercise 3) Verify that the Veronese map $f : \mathbb{RP}^2 \to \mathbb{R}^4$, $[x, y, z] \mapsto (x^2 - y^2, xy, xz, yz)$ is an embedding.

13. (a) Show that D naturally has the structure of a smooth manifold with a covering map $D \to M$ of degree 2. D is called the *orientation double cover* of M.

(b) Show that an orientation of M corresponds to a continuous section of the covering map $D \to M$ (that is, a map $f : M \to D$ such that $f(p) \in D_p$ for each p).

(c) In particular, M is orientable if and only if D is diffeomorphic to the product $M \times \{0, 1\}$ (as a covering space of M).

(d) Show that D is naturally oriented.

14. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that the set of critical values is $\mathbb{Q}$.

15. (a) Prove: if $f : U \subseteq \mathbb{R}^n \to f(U) \subseteq \mathbb{R}^n$ is a diffeomorphism, then

$$\mathcal{L}^n(f(A)) = 0 \Leftrightarrow \mathcal{L}^n(A) = 0$$

for all $A \subseteq U$.

(b) Use **(a)** to construct a consistent definition of "sets of measure zero" in a (second-countable) n-manifold.

16. (a) Describe the group $Isom(T^2)$.

(b) Describe the group $Isom(\mathbb{R}^2)$.

17. Define $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

(a) Prove *exp* is well defined and smooth.

(b) Show $d(\exp(tA))/dt = A\exp(tA)$.

(c) Show $t \mapsto \exp(tA)$ is a 1-parameter subgroup.

(d) Show that $\exp|_U$ is a diffeomorphism onto its image for some open set $U \ni 0$.

(e) Show exp is not in general a local diffeomorphism.

(f) Show $\exp(A) \in GL_+(n, \mathbb{R})$ but $\exp : \mathbb{R}^{n\times n} \to GL_+(n, \mathbb{R})$ is not surjective in general.

18. (The classical Lie Groups) Determine the Lie algebras of the following Lie groups (as a vector space of $n \times n$ matrices) and compute their (real) dimensions:

(a) $GL(n, \mathbb{R}) = \{A \in M^{n\times n}(\mathbb{R}) : \det A \neq 0\}$.

(b) $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$.

(c) $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A^T A = id\}$.

(d) $SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) : \det A = 1\}$.

(e) $GL(n, \mathbb{C})$.

(f) $SL(n, \mathbb{C})$.

(g) $U(n) = \{A \in GL(n, \mathbb{C}) : A^* A = id\}$.

(h) $SU(n) = \{A \in U(n, \mathbb{C}) : \det A = 1\}$.

(i) $Sp(n) = \{A \in GL(2n, \mathbb{Q}) :$ A preserves the standard quaternionic hermitian form$\}$.

(j) Which of these Lie groups are compact, connected or have non-trivial center?