Serie 1

1. Prove a special case of Benford's Law: Let $\{l_n\} = \{l(2)_n\} = \{1, 2, 4, 8, 1, 3, 6..\}$ be the sequence of first digits of the decimal representation of the sequence $\{2^n\} = \{1, 2, 4, 8, 16, 32, 64, ..\}$. Then the density

$$\lim_{N\to\infty} \frac{1}{N} \#\{n : n \leq N, l_n = k\}$$

of an integer $k \in \{1, 2, ..., 9\}$ equals $\log_{10} \frac{k+1}{k}$.

Hint: Use $\log_{10}$ to analyse the first digit of 2^n and the fact (script example 1.9) that $n\alpha \mod \mathbb{Z}$ equi-distributes in $\mathbb{T}$ if α is irrational.

2. Consider the one-dimensional line $\{x_0 + tv \mod \mathbb{Z}^2 : t \in \mathbb{R}\}$ inside the 2-torus $\mathbb{T}^2$, where $x_0 \in \mathbb{T}^2$ and $v = (\alpha, \beta) \in \mathbb{R}^2$. Prove that this line equidistributes, in the sense that

$$\frac{1}{T} \int_0^T f(x_0 + tv \mod \mathbb{Z}^2) dt \to \int_{\mathbb{T}^2} f(x) dx$$

as $T \to \infty$ for any $f \in C(\mathbb{T}^2)$ if and only if α and β are independent over $\mathbb{Q}$.

3. Let $a < b$ be real numbers, and assume that $p \in C^1([a, b])$ and $q \in C([a, b])$ are real-valued functions. We define the second order differential operator

$$L(f) = (pf')' + qf.$$

Also let $\alpha_1, \alpha_2, \beta_1, \beta_2$ and define the boundary conditions

$$\begin{cases} B_1(f) = \alpha_1 f(a) + \alpha_2 f'(a) = 0 \\ B_2(f) = \beta_1 f(b) + \beta_2 f'(b) = 0 \end{cases}$$

Assume that f and g are fundamental solutions, i.e. they form a basis of all solutions $L(f) = 0$, and satisfy additionally $B_1(f) = B_2(g) = 0$ (and therefore also $B_1(g) \neq 0 \neq B_2(f)$). Finally, define the Green function

$$G(s,t) = \begin{cases} cf(s)g(t) & \text{for } a \leq s \leq t \leq b \\ cf(t)g(s) & \text{for } a \leq t \leq s \leq b. \end{cases}$$

a) Show that $p(fg' - f'g)$ is constant.

b) Show that for $h \in C([a, b])$ the boundary-value problem

$$\begin{cases} B_1(u) = B_2(u) = 0 \\ L(u) = h \end{cases}$$

is equivalent to the equation

$$u(s) = K(h)(s) = \int_a^b G(s,t)h(t)dt$$

if the constant c in the definition of G is correctly chosen.

Bitte wenden!

4. Assume that for a region $\Omega \subset \mathbb{R}^d$ any sufficiently *nice* function $f : \Omega \to \mathbb{R}$ can be decomposed into a sum $f = \sum_n f_n$ of functions $f_n : \overline{\Omega} \to \mathbb{R}$ satisfying $\Delta f_n = \lambda_n f_n$ for some $\lambda_n < 0$ and $f_n|_{\partial\Omega} = 0$. Use the discussion of chapter 1.4.1 on the *heat equation* (and fill in the omitted details if possible) to produce a general procedure to solve the boundary value problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times [0, \infty) \\ u|_{\partial\Omega\times\{t\}} = b & \text{for all } t > 0 \\ u|_{\Omega\times\{0\}} = f & \end{cases}$$

where f and b are supposed to be sufficiently nice. No rigorous proof is expected but note where the cheats appear.