Serie 10

An invitation for Weyl's law on graphs. Let M be a manifold and Δ be a Laplacian with eigenvalues λ_n on M. Then Weyl's law (if it holds), named after former ETH professor Hermann Weyl, gives an asymptotic of the type

$$\Lambda(N) := |\{|\lambda_n| \leq N\}| \sim c_d \operatorname{vol}(M) N^{\frac{d}{2}}$$

where d is the dimension of M and c_d some constant only depending on d. Given any finite directed graph $G = (V, E)$ with vertex set V and edge set E, let us identify each edge $e \in E$ with a closed interval $[a_e, b_e]$ endowed with the usual metric on $\mathbb{R}$, glued together in the obvious way given by the graph structure. Thus, we may consider the corresponding function space $L^2(G) = \bigoplus_{e\in E} L^2(a_e, b_e)$. Define a Sobolev space $H^1_{\text{Project}}(G)$ on which $\Delta = \frac{d^2}{dx^2}$ has an orthogonal basis of eigenvectors with eigenvalues satisfying Weyl's law - here $d = 1$, $c_1 = \pi^{-1}$ and $\operatorname{vol}(G) = \sum_{e\in E} |b_e - a_e|$. For the space of functions with Dirichlet's boundary condition $H^1_0(G) = \bigoplus_{e\in E} H^1_0(a_e, b_e)$, exercise 4 of this serie may give a solution. Your goal is to think of a new Sobolev space with different (more interesting) boundary assumption and provide Weyl's law for it!

1. Show that there exists a sequence of compact operators K_n on a Banach space B and a non-compact operator K such that $K_n v \to Kv$ for all $v \in B$.

2. Consider the integral operator $L^2(0,1) \to L^2(0,1)$

$$I(f)(x) = \int_0^x f(t)dt.$$

a) Show that I is compact iff $K : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $K(x_n)_{n\geq 1} = (\frac{x_n}{n})_{n\geq 1}$ is compact and show this for either operator.

b) Calculate the adjoint I^* of I where I^* is defined to be the unique bounded operator $L^2(0,1) \to L^2(0,1)$ for which $\langle If, g\rangle = \langle f, I^*g\rangle$ for all f and g in $L^2(0,1)$.

c) The spectrum of I is defined to be

$$\sigma(I) = \{\lambda \in \mathbb{C} : \lambda \operatorname{Id} - I \text{ is not invertible}\}.$$

Consider the Neumann series of $\lambda \operatorname{Id} - I$ to show that $\sigma(I) \subset \{0\}$.

d) Show that 0 is however no eigenvalue of I, i.e. there does not exists a non-zero function $f \in L^2(0,1)$ such that $I(f) \equiv 0$.

3. We now turn our attention to the Sobolev spaces on the 1-dimensional torus $\mathbb{T}$. The following example will help us to explain how the order of a singularity of a function effects the question whether it lies in $H^k(\mathbb{T})$.

Please turn over!

a) Let $f \in C(\mathbb{T})$ such that f is smooth but for a finitely many points so that it makes sense to consider $f_1 = f' \in L^2(\mathbb{T})$. Let J_ε be a mollifier on $\mathbb{R}$ and show that $f \mapsto (f * J_\epsilon, f_1 * J_\epsilon)$ is a map from $L^2(\mathbb{T})$ to $H^1(\mathbb{T})$ and deduce that f defines an element in $H^1(\mathbb{T})$.

b) Assume that $f \in C(\mathbb{T})$ is identical to $|x|^\alpha$ for some $\alpha \in \mathbb{R}$ on a neighbourhood O of 0 and smooth outside of O. Prove that $f \in H^k(\mathbb{T})$ if $k < \alpha + \frac{1}{2}$.

4. **a)** Show the following description of $H^1(U)$ and $H_0^1(U)$ for $U = (0, 1)$.

$$\begin{aligned} H^1(U) &= \{f \in C(\overline{U}) : f(x) = f(0) + \textstyle\int_0^x g(t)dt \text{ for some } g \in L^2(U)\} \\ H_0^1(U) &= \{f \in H^1(U) : f(0) = f(1) = 0\} \end{aligned}$$

b) Show that any $f \in H^1(U)$ is $\frac{1}{2}$-Hölder continuous meaning that for all $x, y \in \overline{U}$

$$|f(x) - f(y)| \leq c_f |x - y|^{\frac{1}{2}}$$

for some constant c_f.

c) The differential operator $\frac{d}{dx}$ can be continued to a bounded operator $D : H^1(U) \to L^2(U)$. Check that D is in fact a composition of an embedding and a projection $H^1 \hookrightarrow L^2(U)^{\oplus 2} \twoheadrightarrow L^2(U)$. Let D_0 be the restriction of D to $H_0^1(U)$. Compute the adjoint $D_0^* : L^2(U) \to H_0^1(U)$ where the scalar product on $H_0^1(U)$ is that of $L^2(U)^{\oplus 2}$.

d) Identify $H_0^1(U)$ as subspace of $L^2(U)$ endowed with the new scalar product $\langle f, g \rangle_1 = \langle Df, Dg \rangle_{L^2}$ which induces a norm equivalent to $\| \cdot \|_{H_0^1}$. Let $\iota : H_0^1(U) \to L^2(U)$ be the embedding $f \mapsto f$. Compute the adjoint ι^* with respect to this new scalar product. What are the eigenvalues of $\iota \circ \iota^*$? Can you show that $\Delta \circ (-\iota \circ \iota^*)$ is the identity map on $L^2(U)$ where $\Delta = \frac{d^2}{dx^2}$ whenever it makes sense?

e) Does Weyl's law hold for $H_0^1(U)$?