Serie 11

1. Let H be a separable Hilbert space, and $K : H \to H$ a compact and positive operator, i.e. K is self-adjoint and satisfies $\langle Kx, x\rangle \geq 0$ for all $x \in H$. Show that there is a compact, positive operator $\sqrt{K} = S : H \to H$ such that $S \circ S = K$.

Hint: To show that S is compact, show the following type of inverse statement of the Spectral Theorem: Any operator that possesses a system of countable orthonormal eigenvectors that span H such that the corresponding eigenvalues λ_i converge to 0, must be compact.

2. **a)** Recall that we denote for a Banach space Z the space of continuous linear functionals by Z^*. For X and Y Banach spaces, let $F : X \times Y \to \mathbb{C}$ be a bilinear map such that $F(\cdot, y) \in X^*$ and $F(x, \cdot) \in Y^*$ for all $y \in Y$ and $x \in X$ respectively. Show that $F(\cdot, \cdot)$ defines a bounded bilinear operator with respect to the norm

$$\|F\| = \sup_{\|x\|,\|y\|\leq 1} |F(x, y)|.$$

b) Show that any self-adjoint operator must be bounded: Let $A : \mathcal{H} \to \mathcal{H}$ be a linear map on a Hilbert space $\mathcal{H}$ with the self-adjoint property:

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad \forall x, y \in \mathcal{H}.$$

Show that A is continuous.

3. Let $\mathcal{T}$ be a $p + 1$ regular tree with vertices V. By $p + 1$ regular we mean that every vertex has exactly $p + 1$ neighbors. We write $v \sim_k w$ to denote that w is a neighbor of v with distance $k \in \mathbb{N}$, so that the unique path that connects v with w consists of exactly k edges. Since the set of vertices V is countable, we may identify $l^2(V) = l^2(\mathbb{N})$. Let $T_p : l^2(V) \to l^2(V)$ be the averaging operator defined by

$$T_p f(v) = \frac{1}{p+1} \sum_{w\sim v} f(w),$$

where a function $f : V \to \mathbb{C}$ in $l^2(V)$ is evaluted over all distance one neighbors w of v (denoted by $w \sim v$) and let $S_n : l^2(V) \to l^2(V)$ denote the averaging operation over all neighbors with distance less than k:

$$S_n f(v) = \sum_{\substack{v\sim_k w\\ 0\leq k\leq n}} f(w).$$

a) Prove that T_p is a self-adjoint bounded operator. Give a bound of $\|T_p\|$.

b) Let $e_0 = \delta_{v_0}$ denote the basis vector in $l^2(V)$ supported on some fixed vertex $v_0 \in V$. Calculate $\|S_n e_0\|_2$.