Serie 13

1. Consider the following problems related to the Hahn-Banach theorem.

a) Show that any V finite dimensional subspace of a real normed vector space space X is complemented.

b) Proof that if the dual space X^* of a real normed vector space space X is strictly convex (that is, $\|x + y\| = \|x\| + \|y\|$ iff x and y are linearly dependent) then the Hahn-Banach extension of a continuous functional on a subspace on X is unique.

c) Give an explicit example where uniqueness of the Hahn-Banach theorem fails.

d) Assume X to be a normed vector space over $\mathbb{R}$. Assume that dual space X^* is separable and let $\{x_n^*\} \subset X^*$ be countable dense set. Find for each x_n^* find a unit vector $x_n \in X$ such that $x_n^*(x_n) \geq \frac{\|x_n^*\|}{2}$ and verify that $X = \overline{\operatorname{span}_{\mathbb{Q}}\{x_n\}}$.

2. Let $p \in (1, \infty)$ and suppose that q is conjugate to p, that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

a) Prove that $(\ell^p)^* = \ell^q$. Use Hölder's inequality, which says that $a \in \ell^p$ and $b \in \ell^q$ satisfy $\sum |a_n b_n| \leq \|a\|_p \|b\|_q$

b) Prove that a sequence (f_n) converges weakly to a point f in ℓ^p if and only if there exists a constant M such that $\|f_n\|_p \leq M$ and for all $i \in \mathbb{N}$, $f_n(i) \to f(i)$ as $n \to \infty$.

c) Find a sequence in ℓ^p converging weakly to 0, but not converging to 0 in norm.