Serie 2

1. For a closed subspace $W \subset V$ we introduced the norm on the quotient space $X = V/W$:

$$\|v + W\|_X = \inf_{w \in W} \|v + w\|_V.$$

Let $V = C([-1,1], \|\cdot\|_\infty)$ and $W = \left\{ f \in C([-1,1]) : \int_{-1}^0 f(x) d\mu(x) = \int_0^1 f(x) d\mu(x) = 0 \right\}$.

a) Show that W is a closed subspace.

b) Let $f(x) = x$. Show that $\|f\|_{C([-1,1])/W} = \frac{1}{2}$.

c) Show that the infinum is not achieved, i.e. there is no continuous function $g \in W$ such that $\|f + g\|_X = \frac{1}{2}$.

2. Let V be a Banach space and define $V^{\mathbb{N}} = \{a = (a_0, a_1, a_2, a_3, \dots) : a_i \in V\}$ the vector space of sequences in V.

a) Show that if d' is a metric then so is $\frac{d'}{1+d'}$.

b) Show that

$$d(x,y) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on $V^{\mathbb{N}}$ such that $(V^{\mathbb{N}}, d)$ is complete.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. Recall that $C(\overline{\Omega}, \mathbb{R})$ is the space of continuous functions on Ω that can be continuously extended to its boundary $\partial\Omega$. Define

$$C_0(\Omega) = \left\{ f \in C(\overline{\Omega}, \mathbb{R}) : f|_{\partial\Omega} = 0 \right\}.$$

a) Let $f \in C(\overline{\Omega}, \mathbb{R})$ and show that $f \in C_0(\Omega)$ if and only if $|f|^{-1}(\{y \geq \epsilon\}) \subset \Omega$ is compact for any $\epsilon > 0$.

b) Show that the closure of $C_c(\Omega)$ in the space of bounded functions on Ω with respect to the supremum norm is $C_0(\Omega)$. Be aware that you need to show two things: The closure of $C_c(\Omega)$ is contained in $C_0(\Omega)$ and that $C_c(\Omega)$ is dense in $C_0(\Omega)$. Recall that the space of compactly continuous functions $C_c(\Omega)$ is the space of continuous functions $f : \Omega \to \mathbb{R}$ such that the support $\overline{\{x \in \Omega : f(x) \neq 0\}}$ is compact in Ω.

c) Show that $C_0(\Omega)$ is complete.

Please turn over!

4. By standard techniques of the theory of ODE it can be shown that $f(x) = \cos x + \int_0^x \sin(x-t)g(t)dt$ solves the initial value problem on $I = [0,1]$

$$f''(x) + f(x) = g(x), \quad f(0) = 1, f'(0) = 0.$$

For the *Volterra Equation*

$$f''(x) + f(x) = \sigma(x) f(x)$$

this becomes an integral equation. This suggests to define the *integral operator*

$$K : f \mapsto \left(x \mapsto \int_0^x k(x,t) f(t) dt \right)$$

where we have put $k(x,t) = \sigma(x)\sin(x-t)$. If $u(x) = \cos x$, the Volterra equation takes the form $f = u + K(f)$ or, equivalently

$$(\mathrm{Id} - K) f = u.$$

Thus if we can make sense of $(\mathrm{Id} - K)^{-1}$ this inverse operator can be applied to u to get the solution (see chapter 1.3.2 for details).

a) Suppose that $k \in C(I \times I)$ is continuous. Prove that $K : C(I) \longrightarrow C(I)$ as above defines a continuous linear operator. Note that by linearity you only need to prove continuity at the point $f = 0$. Don't forget to show that K actually maps to $C(I)$.

b) The space of continuous linear operators $B = B(C(I), C(I))$ is a normed space with

$$\|K\| = \sup_{\|f\| \le 1} \|K(f)\|.$$

Show that $\|K^n\| \le \frac{\|k\|_\infty^n}{n!}$.

Hint: Use induction on n with the inductive hypothesis that $|(K^n f)(x)| \le \frac{x^n}{n!} \|k\|_\infty^n \|f\|_\infty$.

c) Since B is a Banach space, as you will see in the lecture by next week, prove that $(\mathrm{Id} - K)^{-1} = \sum_{n=0}^\infty K^n$ is well defined.