Serie 3

1. Consider the space of real polynomials $\mathbb{R}[X]$. Define the family of norms

$$\|p\|_s := \|p\|_{C([0,s])} = \sup_{x\in[0,s]} |p(x)|$$

for any $s \in \mathbb{R}_{>0}$. Use the Stone-Weierstrass theorem to show that these are all inequivalent to each other.

2. Let

$$c_c = \{x = (x_j) \in l^\infty(\mathbb{R}) : \#\{j \in \mathbb{N} : x_j \neq 0\} < \infty\}\}$$

be the sequences on $\mathbb{R}$ of compact support endowed with the supremum norm. Show that c_c is not complete. Describe the completion!

3. Consider the two subsets of $l^1(\mathbb{R}) = \{x = (x_j) : \|x\|_1 = \sum_j |x_j| < \infty\}$ given as such

$$A = \{x \in l^1(\mathbb{R}) : x_{2j} = 0 \; \forall j \in \mathbb{N}\} \text{ and } B = \{x \in l^1(\mathbb{R}) : x_{2j-1} = jx_{2j} \; \forall j \in \mathbb{N}_{>0}\}.$$

Show that both A and B are closed $l^1(\mathbb{R})$ (with respect to $\|\cdot\|_1$) but $A + B = \{x + y : x \in A, y \in B\}$ is not!

4. Characterize the compact subsets of the following Banach spaces.

1. The space c_0 of null sequences (that is, sequences (x_n) of scalars with $|x_n| \to 0$ as $n \to \infty$) with the norm $\|(x_n)\|_\infty = \sup_{n\geq 1} |x_n| = \max_{n\geq 1} |x_n|$.
2. The space l^p of p-summable sequences of scalars, that is,

$$l^p = \{(x_n) | \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

with the p-norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}.$$

3. The space $C_0(X)$ of continuous functions vanishing at infinity with the uniform norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$, where X is a separable locally compact metric space. A function f is said to vanish at infinity if for any $\varepsilon > 0$ there exists a compact subset $K \subset X$ for which $\sup_{x\in K^c} |f(x)| < \varepsilon$ (cf. Example 2.19(4)). The proof of the Arzela-Ascoli theorem will help you!