Serie 4

1. **a)** Let X and Y be two compact spaces. Proof that the space of functions

$$E = \left\{ \sum_{l=1}^{n} f_l g_l : f_l \in C(X), g_l \in C(Y), n \in \mathbb{N} \right\}$$

is dense in $C(X \times Y)$.

b) Extend the Stone-Weierstrass theorem for $C_0(X)$ where X is a locally compact Hausdorff space.

2. Prove that for any open $\Omega \subset \mathbb{R}^n$ the set $C_c^\infty(\Omega)$ of smooth functions with compact support is dense in $L^1(\Omega, \lambda)$ where λ is the usual Lebesque measure.

a) Define $J(x) = ke^{\frac{-1}{1-|x|^2}}$ for $|x| < 1$ and equal to zero elsewhere. Here, the constant k is chosen such that $\int_{\mathbb{R}^n} J = 1$. Prove that the *mollifier* $J_\epsilon(x) = \frac{1}{\epsilon^n} J(\frac{x}{\epsilon})$ vanishes for $|x| \geq \epsilon$ and $\int_{\mathbb{R}^n} J_\epsilon = 1$.

For $f \in L^1(\Omega)$ define the *regularization* of f by convolving with J_ϵ:

$$f_\epsilon(x) = J_\epsilon * f(x) = \int_\Omega J_\epsilon(x-y) f(y) d\lambda(y).$$

b) Prove that f_ϵ is integrable.

c) Prove that f_ϵ is smooth.

d) Prove that if f has compact support then so does f_ϵ.

e) Use the fact that $C_c(\Omega)$ is dense in $L^1(\Omega)$ (Proposition 2.39) to finish the proof: For any $f \in L^1(\Omega)$ there exists $g \in C_c^\infty(\Omega)$ such that $\|f - g\| < \epsilon$.

3. Chapter 1.5 gives a short introduction to the Dirac function δ as a principal example for a distribution (= a linear continuous map from a space of *test functions* (e.g. $\mathcal{D} = C_c^\infty(\mathbb{R})$) to $\mathbb{R}$). It is defined by the following property: $\delta(\phi) = \phi(0)$ for any $\phi \in \mathcal{D}$.

a) Prove that there is no $F \in L^1_{\text{loc}}(\mathbb{R})$ that satisfies $\int_{\mathbb{R}} F(x)\phi(x) d\lambda(x) = \phi(0)$ for any $\phi \in \mathcal{D}$.

b) Use the machinery from of the last exercise to prove that there is a sequence of functions $\{\delta_n\}$ in $C_c^\infty(\mathbb{R})$ that approximates δ in the following sense:

$$\int_{\mathbb{R}} \delta_n(x)\phi(x) d\lambda(x) \to \phi(0) \text{ as } n \to \infty$$

for any $\phi \in \mathcal{D}$. We will say that (the sequence of functionals defined by) δ_n converges *weak** to δ. This notion will appear later in the course.

Please turn over!

4. **a)** Consider once more the linear map $\delta : C([0,1]) \to \mathbb{R}$ given by $\delta(f) = f(0)$. Show that for any $1 \leq p < \infty$, δ is not continuous with respect to the topology induced by the norm

$$\|f\|_p = \left(\int_0^1 |f|^p \, dx \right)^{\frac{1}{p}}.$$

b) For any $a < b$, consider the operator defined by $L : L^2([a,b]) \to \mathbb{R}$ given by

$$L(f) = \int_a^b f(x)\, dx.$$

Show that L is a well-defined, continuous, linear operator and compute $\|L\|$.

c) Consider the shift operator $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ given by

$$S(x) = (x_2, x_3, x_4, \dots).$$

where $x = (x_1, x_2, x_3, \dots)$. Show that $\|S\| = 1$, but S is not an isometry. Does S have any eigenvalues? If so, how many?

d) Let $c_c(\mathbb{N}) \subset l^\infty(\mathbb{N})$ be the space sequences of compact support (see Serie 3) equipped with the supremum norm. Define $T : c_c(\mathbb{N}) \to c_c(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, \dots).$$

Show that T is not continuous. Construct a sequence of linear continuous operators T_k on c_c for which $T_k x \to Tx$ for all $x \in c_c$ in l^∞.