Serie 5

1. Let $D = B_1^{\mathbb{C}}(0)$ be the open unit disc, with its boundary parameterized by the curve $\gamma(t) := \mathrm{e}^{2\pi i t}$, $t \in [0,1)$. Let V be the space of holomorphic functions on D. Fix $p \in [1,\infty)$.

a) Consider the norm on V

$$\|f\|_{A^p(D)} = \|f\|_{L^p(D)}$$

and call

$$\{f \in V : \|f\|_{A^p(D)} < \infty\}$$

the *Bergman space* $A^p(D)$. Show that for all $z \in D$ the linear map $E_z : f \longmapsto f(z)$ is continuous. More generally, show that if $K \subset D$ is compact then

$$A^p(D) \ni f \longmapsto f|_K \in C(K)$$

is a bounded operator with respect to $\|\cdot\|_\infty$ on $C(K)$. Conclude that $A^p(D)$ is a Banach space.

b) Consider now

$$\|f\|_{H^p(D)} = \sup_{0<r<1} \left(\int_0^1 |f(r\gamma(t))|^p \mathrm{d}t \right)^{1/p}.$$

and call

$$\{f \in V : \|f\|_{H^p(D)} < \infty\}$$

the *Hardy space* $H^p(D)$. Show once more that

$$H^p(D) \ni f \longmapsto f|_K \in C(K)$$

is a bounded operator.

Comment: One can show that $\tilde{f}(t) = \lim_{r\to 1} f(r\gamma(t))$ exists for almost all t, $\|\tilde{f}\|_{L^p(\partial D)} = \|f\|_{H^p(D)}$ and that in fact $L^p(\partial D) \simeq H^p(D)$. In particular, $H^p(D)$ is a again a Banach space.

2. Let A be a unital Banach algebra.

a) Show that the subset of invertible elements $A^\times$ is open in A.

b) Show that the "taking inverse" map on $A^\times$, $x \mapsto x^{-1}$ is continuous.

This shows that if V is a Banach space then

$$\mathrm{GL}(V) = \{T \in B(V,V) : T \text{ is bijective and } T^{-1} \text{ is continuous}\}$$

is open with respect to the operator norm. Do you know an easy argument if V is finite dimensional?

3. Compute the operator norm of the continuous map $f \mapsto f$ when viewed as

a) a map from $C^1([0,1])$ with the C^1-norm to $C([0,1])$ with the sup-norm.

Please turn over!

b) a map from $C([0,1])$ with the sup-norm to $L^1([0,1], dx)$.

c) Compute the operator norm of the composition of the above maps.

d) Restrict the above maps to the subspace of functions f with $f(0) = 0$, and compute the operator norm again.

4. Show that an inner product on an inner product space is jointly continuous with respect to the induced norm: if $\lim_n v_n = v$ and $\lim_n w_n = w$, then $\langle v_n, w_n \rangle \to \langle v, w \rangle$ as $n \to \infty$.